EE 320 Introductory Mathematical Economics

Semester 2/2015

Homework 5

Solution

Question 1:

Given the production function

$$Q = f(K, L) = K^{0.5}L^{0.5}$$

Suppose that the per unit input prices for K and L are \$20 and \$5, respectively.

a. If the producer needs to maintain the output level at $\overline{Q} = 20$ units, what are the values K^* *and* L^* *that minimizes the total cost*, and what is the corresponding minimum cost? Use the bordered Hessian to verify that the second-order sufficient condition is met.

$Min_{K,L}\ TC = 20K + 5L$ subject to $K^{0.5}L^{0.5} = 20$

$L(K, L, \lambda) = 20K + 5L + \lambda[20 - K^{0.5}L^{0.5}]$

FOC:

$L_K = 20 - \lambda(0.5K^{-0.5}L^{0.5}) = 0$ -- (1)

$L_L = 5 - \lambda(0.5K^{0.5}L^{-0.5}) = 0$ -- (2)

$L_\lambda = 20 - K^{0.5}L^{0.5} = 0$ -- (3)

(1), (2), & (3) $\Rightarrow$ $K^* = 10$; $L^* = 40$; $\lambda^* = 20$

$\Rightarrow$ TC* = 400.

SOSC:

$$|\bar{H}_2| = \begin{vmatrix} 0 & 1 & 1/4 \\ 1 & 1 & -1/4 \\ 1/4 & -1/4 & 1/16 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -1/4 \\ 1/4 & 1/16 \end{vmatrix} + \left(\tfrac{1}{4}\right)\begin{vmatrix} 1 & 1 \\ 1/4 & -1/4 \end{vmatrix}$$

$$= -\frac{1}{8} - \frac{1}{8} = -\frac{1}{4} < 0$$

$\therefore$ SOSC for a minimum TC is satisfied.

b. Suppose now that the output level is not fixed, but the producer has a budget constraint at \$400. Assume that this producer spends his entire budget on the production. Determine the values K^* *and* L^* *that maximizes the total output.* Also, use the bordered Hessian to verify that the second-order sufficient condition is met.

$Max_{K,L}\ Q = K^{0.5}L^{0.5}$ subject to $20K + 5L = 400$

$L(K, L, \lambda) = 20K + 5L + \lambda[400 - 20K - 5L]$

FOC:

$L_K = 0.5K^{-0.5}L^{0.5} - 20\lambda = 0$ -- (1)

$L_L = 0.5K^{0.5}L^{-0.5} - 5\lambda = 0$ -- (2)

$L_\lambda = 400 - 20K - 5L = 0$ -- (3)

(1), (2), & (3) $\Rightarrow$ $K^* = 10$ and $L^* = 40$, which are the same as the values found in part (a).

Question 2:

Pakorn's utility function depends on the consumption of two commodities, x and y, and it is given by

$$U(x, y) = 2xy$$

Suppose that his income is \$72, and the prices per unit of x and y are \$4 and \$6, respectively. Assume that Pakorn spends all of his income, and the values of x and y are both non-zero.

a. Use the Lagrange method to determine the values of x^* and y^* that maximize Pakorn's utility given an income constraint. Verify that the second-order sufficient conditions are satisfied.

$$L(x, y, \lambda) = 2xy + \lambda[72 - 4x - 6y]$$

FOC: $L_x = 0 \Rightarrow 2y - 4\lambda = 0$ -- (1)

$L_y = 0 \Rightarrow 2x - 6\lambda = 0$ -- (2)

$L_\lambda = 0 \Rightarrow 72 - 4x - 6y = 0$ -- (3)

(1), (2), and (3) $\Rightarrow x^* = 9,\ y^* = 6,\ \lambda^* = 3$

SOSC:

H = [0 4 6]

[4 0 2]

[6 0 2]

$\Rightarrow |\overline{H}| = 96 > 0$. Thus, the SOSC for a maximum is satisfied.

b. Determine the maximum utility level and the Lagrange multiplier. Interpret the economic interpretation of the Lagrange multiplier.

$U^* = 2(9)(6) = 108$

$\lambda^* = 3$

$\lambda^* = \frac{dU^*}{dY} = 3$. The Lagrange multiplier is the marginal utility of income. It is the rate of change in the maximum utility with respect to the change in income. For instance, if the income changes by 1 unit, the maximum utility would increase approximately by 3 unit.

c. Suppose that the income is now \$73. Approximate the new maximum utility level.

From b., the new maximum utility should be approximately equal to 111.

$$\lambda^* = \frac{dU^*}{dY} \Rightarrow \; dU^* \approx \lambda^* dY = (3)(73-72) = 3$$

$$U^{**} = U^* + \Delta U \approx U^* + dU^* = 108 + 3 = 111.$$

<u>Question 3: Integration</u>

Suppose that a monopolist faces the demand function $Q = 16 - P$. Its total cost function is given by $TC(Q) = 4Q + Q^2$.

a. Suppose that this monopolist cannot price-discriminate. Use integral to calculate the consumer and producer surplus at the profit-maximizing quantity and price

Inverse demand: P = 16- Q. $\Rightarrow \; TR(Q) = 16Q - Q^2$

$$Max_Q \, \pi(Q) = TR(Q) - TC(Q) = (16Q - Q^2) - (4Q + Q^2)$$

$$\text{FOC: } \pi' = 0 \Rightarrow \; MR(Q) - MC(Q) = 16 - 2Q - (4 + 2Q) = 0$$

$$\Rightarrow \quad Q^* = 3$$

$$\Rightarrow \quad P^* = 13$$

$$CS = \int_0^3 [(16 - Q) - 13]dQ = \int_0^3 [3 - Q]dQ = (3Q - \tfrac{Q^2}{2})|_0^3 \; = \; 4.5$$

$$PS = \int_0^3 [13 - (4 + 2Q)]dQ = \int_0^3 [9 - 2Q]dQ = (9Q - \tfrac{2Q^2}{2})|_0^3 \; = \; 18$$

b. If the monopolist can now practice price-discrimination; that is, he can perfectly identify its new profit-maximizing output level. Also, use integral to calculate the consumer and producer surplus at this new quantity, and discuss the change in total welfare.

For perfect price-discriminator, he can charge each consumer the maximum amount he or she is willing to pay. Hence, the profit is maximized where D = MC.

That is, $16 - Q = 4 + 2Q \quad \Rightarrow \quad Q^* = 4$

CS = 0

$$PS = \int_0^4 [(16 - Q) - (4 - 2Q)]dQ = \int_0^4 [12 - 3Q]dQ = 24 .$$

For perfect price discrimination, producer surplus is equal to $24 and consumer has no consumer surplus. Hence, total social welfare is $24.

In the case of monopolist in part a, CS = 4.5 and PS=18, which means that total social welfare is $22.5.

We can see that the total social welfare is higher in the case of perfect price discrimination, but at a cost for consumers.

Question 4: Cost minimization problem

Consider a cost minimization problem where firm chooses for optimal combination of capital (K) and labor (L). Suppose that r and w are the prices per unit of capital and labor, respectively. And assume further that the production technology of this firm is given by $\sqrt{K} + L = Q$. Consider the following problem

a. Solve for the optimal combination of capital and labor.

Setting the LaGrange function and solve for the stationary point. This yields us:

$[K]: \quad r - \lambda \frac{1}{2\sqrt{K}} = 0$

$[L]: \quad w - \lambda = 0 \quad \Rightarrow \quad \lambda = w$

$K^* = (\frac{w}{2r})^2 \quad \Rightarrow \quad L = Q - \frac{w}{2r}$

b. State the condition under which both types of factor inputs are used by firm.

$L = Q - \frac{w}{2r} > 0$ That is, $\frac{w}{2r} < Q$.

Note that if the inequality doesn't hold, it would be optimal for firm to use only capital in the production.

c. Derive the cost function.

$$C = wL + rK = w(Q - \tfrac{w}{2r}) + r(\tfrac{w}{2r})^2$$

Question 5 *Integration*

a. Suppose the demand and supply curves are $P = \frac{6000}{Q+50}$ and $P = Q + 10$. Find the equilibrium price and quantity, and compute the consumer and producer surplus.

Solve for the equilibrium:

$(Q+10)(Q+50) = 6000 \quad \Rightarrow Q = 50. \Rightarrow P = 60.$

$$CS = \int_0^{50} \frac{6000}{Q+50}\, dQ - 60 * 50 = 6000ln(Q+50)]^{50}{}_0 - 3000 = 1158.8$$

PS = (50)(60) - 1750 = 1,250

b. Let $MR = 25 - 5x - 2x^2$ and $MC = 10 - 3x - x^2$, where x is the unit of output. Assume that fixed cost is \$7. Determine the level of production that contributes to maximum profit and determine the level of maximized profit.

Setting MR = MC, and we yield that $x = 3$.

$\pi(x) = R(x) - C(x) = \int (MR - MC)\, dx$

$R(x) = \int MR(x)\, dx = 25x - 5x^2/2 - 2x^3/3 + constant_1$

$R(0) = 0 \Rightarrow constant_1 = 0$

$C(x) = \int MC(x)\, dx = 10x - 3x^2 - x^3/3 + constant_2$

$C(0) = 7 \Rightarrow constant_2 = 7$

Using all everything shown above, we can find that $\pi(3) = 20$.

Question 6

The production function for a company's product is Q = $100L + 50K - L^2 - K^2$, where Q is the output that results from L units of labor and K units of capital. The unit costs of labor and capital are \$6 and \$3, respectively. Consider the following problem

a. If the company wants the total cost of inputs to be 30, determine the greatest output possible subject to this budget constraint.

   $[L]: \quad 100 - 2L - 6\lambda = 0$

   $[K]: \quad 50 - 2K - 3\lambda = 0$

   $[L]: \quad 6L + 3K = 30$

   Solve K and L. The answer is K = 2 and L = 4. ($\lambda = 46/3$)

b. Suppose market price of the product is \$12 per unit. Determine the optimal combination of labor and capital that yields this company the highest level of profit.

   $12(100 - 2L) = 6 \implies L = 199/2$

   $12(50 - 2K) = 3 \implies K = 597/3$