Answer HW 1 EE325

1 a) $$\sum_{i=1}^{6}(a+bx_i) = \sum_{i=1}^{6}a + \sum_{i=1}^{6}bx_i$$
$$= 6a + b\sum_{i=1}^{6}x_i$$

b) $$\sum_{y=0}^{3}f(x+y) = f(x+0)+f(x+1)+f(x+2)+f(x+3)$$

c) $$\sum_{i=1}^{8}i^2 = 1^2+2^2+\ldots+8^2$$

d) $$\sum_{x=2}^{3}\sum_{y=3}^{4}(3x+2y) = \sum_{x=2}^{3}((3x+6)+(3x+8))$$
$$= (3(2)+6)+(3(3)+6)+$$
$$(3(3)+8)+(3(3)+8)$$

2 a) $\sum f(x) = 1$

$= 0.5b + b + 2.25b + 2b + 1.5b + 0.5b + 0.25b$

$= 8b$

$b = \frac{1}{8}$

b) $P(X \leq 3) = 1 - 0.75b = 1 - 0.75\left(\frac{1}{8}\right)$

c) $P(-3 \leq X \leq 3) = 1 - (P(X > 3) + P(X \leq -3))$

$= 1 - (0.25b + 0.5b)$

$= 1 - 0.30b = 1 - 0.30\left(\frac{1}{8}\right)$

d) $P(X \geqslant 1) = 1 - P(X < 1)$

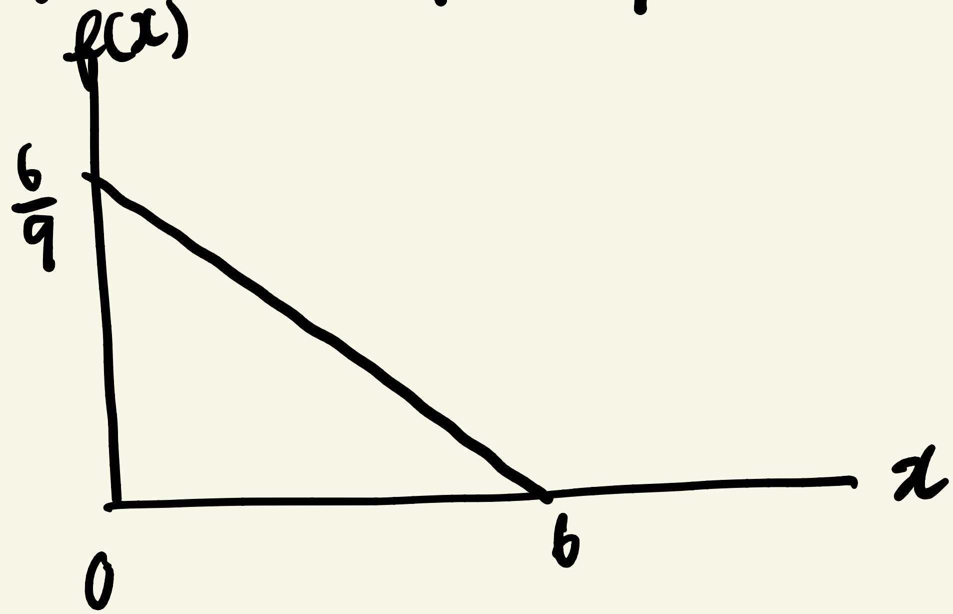
$$= 1 - (0.5b + b + 2.25b + 2b)$$

$$= 1 - 3.75b$$

$$= 1 - 3.75\left(\frac{1}{8}\right)$$

3 $f(x) = -\frac{1}{9}x + \frac{6}{9}, \quad 0 \le x \le 3$

a)



b) $P(1 \le X \le 3)$

$$\int_1^3 f(x)\,dx = \int_1^3 \left(-\frac{1}{9}x + \frac{6}{9}\right) dx$$

$$= -\frac{x^2}{18} + \frac{6x}{9} \Big|_1^3 = \frac{8}{9}$$

c) $P(X \geq 2) = \int_2^3 f(x)\,dx$

$= \int_2^3 \left(-\frac{1}{9}x + \frac{6}{9}\right) dx$

$= \frac{-x^2}{18} + \frac{6x}{9} \Big|_2^3 \, dx = \frac{7}{18}$

d) $E(X) = \int_0^3 x f(x)\,dx$

$= \int_0^3 x\left(-\frac{1}{9}x + \frac{6}{9}\right) dx = \int_0^3 \frac{-x^2}{9} + \frac{6x}{9}\,dx$

$= \frac{-x^3}{27} + \frac{3x^2}{9} \Big|_0^3 = 2$

4 a), b), c)

	X = 1	2	3	4	5	6	$f(y)$
Y = 0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{6}{12}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{6}{12}$
$f(x)$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	

d) $f(X=1 \mid Y=1) = \dfrac{\frac{1}{12}}{\frac{1}{2}} = \dfrac{1}{6}$

$\vdots \qquad\qquad \vdots$

$f(X=6 \mid Y=1) = \dfrac{\frac{1}{12}}{\frac{1}{2}} = \dfrac{1}{6}$

e) $E(X \mid Y=1) = \sum_x x f(x \mid Y=1)$

$= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right)$

$+ 6\left(\frac{1}{6}\right) = \dfrac{21}{6}$

f) $\mathrm{Var}(X|Y=1) = \sum_x (x - E(X|Y=1))^2 f(x|Y=1)$

$$= \left(1-\frac{21}{6}\right)^2\left(\frac{1}{6}\right) + \left(2-\frac{21}{6}\right)^2\left(\frac{1}{6}\right) + \ldots + \left(6-\frac{21}{6}\right)^2\left(\frac{1}{6}\right)$$

$$= \frac{35}{12}$$

5) $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$

$$E(\bar{X}) = E\left(\frac{1}{3}\sum_{i=1}^{3} X_i\right)$$

$$= \frac{1}{\cancel{3}} \cancel{3}\, E(X_i)$$

$$= E(X_i) = \mu$$

$$\operatorname{cov}(X_1, X_2) = \operatorname{cov}(X_1, X_3) = \operatorname{cov}(X_2, X_3) = \frac{1}{4}\sigma^2$$

$$\operatorname{var}(\bar{X}) = \frac{1}{9}\operatorname{var}(X_1, X_2, X_3)$$

$$= \frac{1}{9}\Big(\operatorname{Var}(X_1) + \operatorname{Var}(X_2) + \operatorname{Var}(X_3) + 2\operatorname{cov}(X_1, X_2)$$

$$+ 2\operatorname{cov}(X_1, X_3) + 2\operatorname{cov}(X_2, X_3)\Big)$$

$$= \frac{1}{9}\left(\sigma^2 + \sigma^2 + \sigma^2 + 2\left(\frac{1}{4}\sigma^2\right) + 2\left(\frac{1}{4}\sigma^2\right) + 2\left(\frac{1}{4}\sigma^2\right)\right)$$

$$= \frac{1}{9}\left(3\sigma^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2\right) = \frac{1}{9}\left(3\sigma^2 + \frac{3}{2}\sigma^2\right)$$

$$= \frac{1}{9}\left(\frac{6\sigma^2 + 3\sigma^2}{2}\right) = \frac{1}{\cancel{9}}\left(\frac{\cancel{9}\sigma^2}{2}\right) = \frac{\sigma^2}{2}$$

6 a) $E(\bar{X}) = E\left(\frac{1}{4}\sum_{i=1}^{4} X_i\right) = \frac{1}{\cancel{4}}\cancel{4}\,\mu = \frac{1}{\cancel{4}}\cancel{4}E(X_i)$

$$= \mu = E(X_i)$$

$$var(\bar{X}) = var\left(\frac{1}{4}(X_1+X_2+X_3+X_4)\right)$$

$$= \frac{1}{16} var(X_1+X_2+X_3+X_4)$$

$$= \frac{1}{16}(var(X_1)+var(X_2)+var(X_3)+var(X_4))$$

$$= \frac{1}{16}(\sigma^2+\sigma^2+\sigma^2+\sigma^2) = \frac{4\sigma^2}{16} = \frac{\sigma^2}{4}$$

b) $\tilde{X} = \frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4$

$$E(\tilde{X}) = E\left(\frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4\right)$$

$$= \frac{1}{8}E(X_1) + \frac{1}{4}E(X_2) + \frac{1}{8}E(X_3) + \frac{1}{2}E(X_4)$$

$$= \frac{1}{8}\mu + \frac{1}{4}\mu + \frac{1}{8}\mu + \frac{1}{2}\mu$$

$$\frac{\mu + 2\mu + \mu + 4\mu}{8} = \mu$$

$\tilde{X}$ is unbiased estimator of μ.

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c) $\text{Var}(\tilde{X}) = \text{var}\left(\frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4\right)$

$$= \text{var}\left(\frac{X_1 + 2X_2 + X_3 + 4X_4}{8}\right)$$

$$= \frac{1}{64}\text{var}(X_1 + 2X_2 + X_3 + 4X_4)$$

$$= \frac{1}{64}\left(\text{var}(X_1) + 4\,\text{Var}(X_2) + \text{Var}(X_3) + 16\,\text{Var}(X_4)\right)$$

$$= \frac{1}{64}\left(\sigma^2 + 4\sigma^2 + \sigma^2 + 16\sigma^2\right) = \frac{22\sigma^2}{64} = \frac{11}{32}\sigma^2$$

$$\text{var}(\bar{X}) < \text{var}(\tilde{X})$$

$\therefore$ $\bar{X}$ is better estimator for μ.