Solution: Quiz 6

1. Consider a function
$$f = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \,\middle|\, \quad y = 3x^5 - 5x^3\right\}.$$
Identify x and y intercepts, asymptotes, relative extrema, point of inflection (if any), and concavity of f. Sketch the curve of f.

Note: $f(x) = 3x^5 - 5x^3$, $f'(x) = 15x^4 - 15x^2$, $f''(x) = 60x^3 - 30x$, $\frac{1}{\sqrt{2}} \approx 0.7071$.

Solution:

(i) Find intercepts and points of inflection:

- The y-intercept(x=0) is at $y = f(0) = 6(0)^5 - 10(0)^3 = 0 \Rightarrow \; y = 0$.
- The x-intercept(y=0) occurs when $0 = f(x) = x^3(3x^2 - 5)$
$\Rightarrow$ at $\; x = 0,\ x = -\sqrt{5/3},\ x = \sqrt{5/3}$.

(ii) To identify the asymptote, we can view the function $f(x) = 3x^5 - 5x^3$ as a rational function $f(x) = \frac{p(x)}{q(x)}$ where $p(x) = 3x^5 - 5x^3$ has degree $m = 5$ and $q(x) = 1$ has degree $n = 0$. Since $q(x) = 1$ is not zero for any x, there is no vertical asymptote. Since $m \nleq n$ and $m \neq n+1$, there is no horizontal or slant asypmtote.
(iii) Relative Extrema: To find critical numbers,

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x - 1)(x + 1) = 0 \;\Rightarrow$$

$x = 0, -1, 1$ are the only critical numbers. From $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$ and from the Second Derivative Test,
$f''(0) = 0 \Longrightarrow$, no conclusion
$f''(-1) = -30 < 0 \Longrightarrow$, $f(-1) = -3 + 5 = 2$ is a relative maximum
$f''(1) = 30 > 0 \Longrightarrow$, $f(1) = 3 - 5 = -2$ is a relative minimum.

The First Derivative Test will be used to identify the critical number $x = 0$ as shown in the table. Since the sign of $f'(x)$ does not change at $x = 0$, $x = 0$ does not give an extremum.

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1)$	$x \in (1, \infty)$
Sign of $f'(x) = 15x^2(x-1)(x+1)$	$+$	$-$	$-$	$+$
$x^2(x-1)(x+1)$	$(+)(-)(-)$	$(+)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

(iv) To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 30x(2x^2 - 1) = 0 \qquad \Longrightarrow \quad x = 0, -\sqrt{1/2}, \sqrt{1/2}.$$

	$x \in (-\infty, -\sqrt{\frac{1}{2}})$	$x \in (-\sqrt{\frac{1}{2}}, 0)$	$x \in (0, \sqrt{\frac{1}{2}})$	$x \in (\sqrt{\frac{1}{2}}, \infty)$
Sign of $f''(x) = 30x(2x^2 - 1)$	$-$	$+$	$-$	$+$
$(x)(x - \sqrt{1/2})(x + \sqrt{1/2})$	$(-)(-)(+)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{1/2}, \sqrt{1/2}$, they give inflection points. Since $f(0) = 0$, $f(-\sqrt{1/2}) = 7\sqrt{2}/8$, $f(\sqrt{1/2}) = -7\sqrt{2}/8$, the inflection points are $(0,0)$, $(-\sqrt{1/2}, 7\sqrt{2}/8)$,$(\sqrt{1/2}, -7\sqrt{2}/8)$.

- From the table in (iii), we see that
$f(x)$ is increasing ($f'(x) > 0$) on $(-\infty,-1) \cup (1,\infty)$ and
$f(x)$ is decreasing ($f'(x) < 0$) on $(-1,1)$.

(v) From the table in (iv), we see that
$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{\frac{1}{2}}, 0) \cup (\sqrt{\frac{1}{2}}, \infty)$ and
$f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{\frac{1}{2}}) \cup (0, \sqrt{\frac{1}{2}})$.