Mean-Variance Analysis and the Pricing of Risk (Cont.)

EE431/438

Copeland, Thomas E. and J. Fred Weston, Financial Theory and Corporate Policy (4th ed), Addison-Wesley, 2005: Ch5 (pp 101 -141)

September 2011

Contents

- Last example: consider a case where the returns of the two risky assets had a negative correlation
- Now we are going to consider the two cases ; (1) perfectly correlated $r_{xy} = 1$ (2) perfectly inverselt correlated $r_{xy} = -1$
- $Y = a + bX \Rightarrow b$ is a positive number , X and Y are perfectly correlated $r_{xy} = 1$
- Proof : $Y = a + bX$
  $E(Y) =$ ....
  $VAR(Y) =$ ....
  $\sigma_y =$ ....
  $$r_{xy} = \frac{COV(X, Y)}{\sigma_x \sigma_y} = \frac{E(X - E(X))(Y - E(Y))}{\sigma_x \sigma_y}$$
  $$r_{xy} =$$
  $$=$$

- $Y = a - bX \Rightarrow b$ is a positive number , X and Y are perfectly inversely correlated $r_{xy} = -1$
- Proof : $Y = a - bX$
  $E(Y) =$ ....
  $VAR(Y) =$ ....
  $\sigma_y =$ ....
  $$r_{xy} = \frac{COV(X,Y)}{\sigma_x \sigma_y} = \frac{E(X - E(X))(Y - E(Y))}{\sigma_x \sigma_y}$$
  $$r_{xy} =$$
  $$=$$

Perfectly Correlated Assets (Cont 2)

- Example

p_i=probabilty	X(%)	Y(%)
0.2	-1.408	-3
0.2	17.258	15
0.2	3.777	2
0.2	22.443	20
0.2	7.925	6
1		

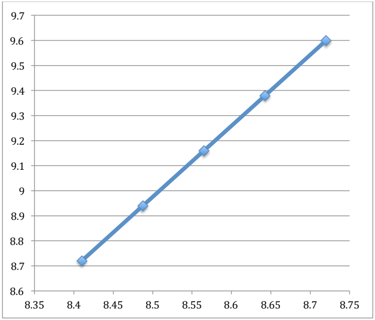
- $X = 1.037Y + 1.703$
- $E(Y) = 8\%$, $\sigma_y = 8.41\%$
- $E(X) =$ ....
  $= 9.599$
- $VAR(X) =$ ....
  $\sigma_x = 8.72\%$
-
$$\begin{aligned} COV(X, Y) &= \\ &= 0.007334 \end{aligned}$$

Perfectly Correlated Assets (Cont 3)

- A portfolio consisting X and Y :
- $E(R_p) =$ ....
- $VAR(R_p) =$ ....
- $\sigma_p =$

- The relation between the portfolio return and the portfolio standard deviation

$E(R_P)$

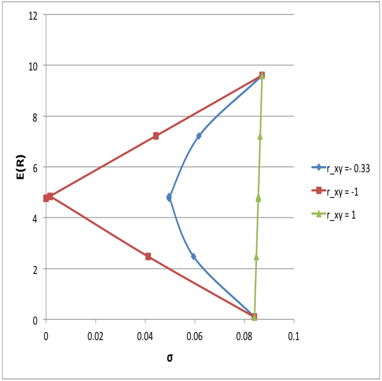


σ_p

- It can be mathematically proven that the curve between A and B is a straight line
- Slope $= \dfrac{dE(R_p)}{d\sigma(R_p)} =$
- $E(R_p) = aE(X) + (1-a)E(Y)$; $\dfrac{dE(R_p)}{da} =$
- $\sigma(R_p) = a\sigma_x + (1-a)\sigma_y$; $\dfrac{d\sigma(R_p)}{da} =$
- $\text{Slope} = \dfrac{dE(R_p)}{d\sigma(R_p)} =$
- The slope is a constant. It does not depend on a. Therefore, AB is a straight line.

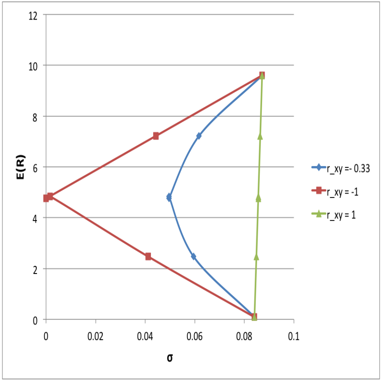
- Suppose that the returns on X and Y are perfectly inversely correlated; $r_{x,y} = -1$
- A portfolio consisting X and Y :
- $E(R_p) =$ ....
- $VAR(R_p) =$ ....
- $\sigma_p =$

Perfectly Correlated Assets (Cont 7)



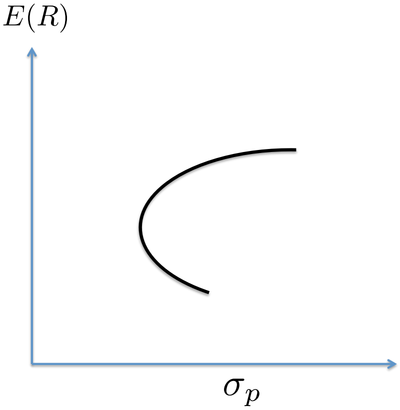
- If $r_{xy} = -1$, the minimum variance portfolio has zero variance
- To prove,
  - calculate $a^* = \dfrac{\sigma_y^2 - r_{xy}\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2 - 2r_{xy}\sigma_x\sigma_y}$
  - calculate E(R) and σ^2 of the minimum variance portfolio
- The relationship between E(R) and σ is as shown in the figure.

Minimum Variance Opportunity Set



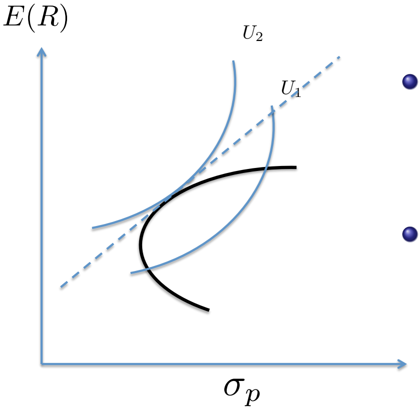
- Usually assets are less than perfectly correlated ; $-1 < r_{xy} < 1$
- The general slope of the mean-variance opportunity set is convex.
- Minimum vraince opportunity set : portfolio in the opportunity set which yields minimum variance for a given rate of return

The Efficient Frontier with Two Risky Assets (No Risk-Free Asset)



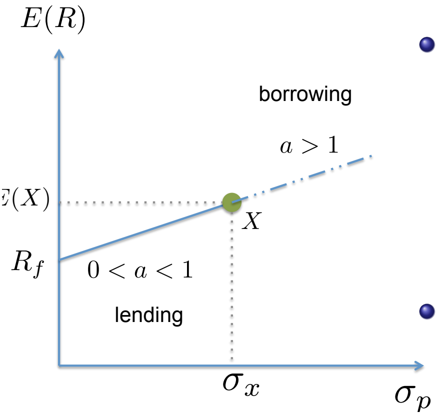
- utility maximising portfolio is the one where indifference curve is tangent to the minimum variance portfolio set
- different investors may hold different portfolios
- (later when riskless asset is introduced, investors will hold identical combinations of risky assets)

The Efficient Frontier with Two Risky Assets (No Risk-Free Asset)



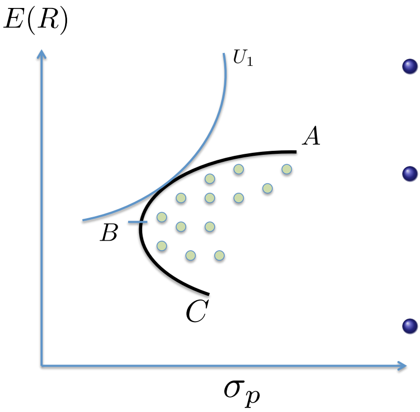
- rational investors will never choose a portfolio below the minimum variance point
- Portfolios above the minimum variance point dominate those below the point
- Efficient frontier; portfolios which offer higher return than those of an equivalent level of risk or alternatively, entail less risk at an equivalent level of return.
- first developed by Harry Markowitz

The Efficient Frontier with One Risky Asset and One Risk-Free Asset

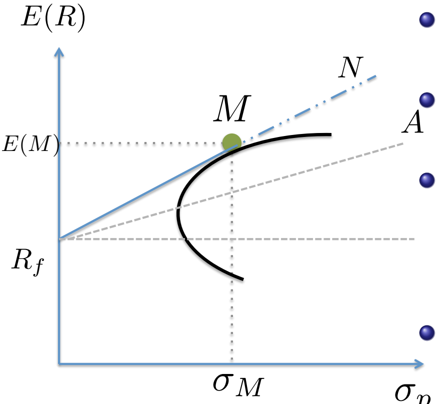


- Risk-free asset, R_f, has zero variance, then the mean and the variance of the portfolio become
  - $E(R_p) = aE(X) + (1 - a)R_f$
  - $VAR(R_p) = a^2 VAR(X)$
- The relationship between expected return and standard deviation is linear

Optimal Portfolio Choice: N risky Assets

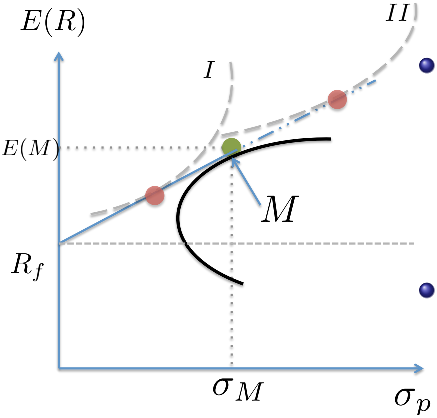


- Portfolio Mean ;
$$E(R_p) = \sum_{i=1}^{n} w_i E(R_i)$$
- Portfolio Variance ;
$$VAR(R_p) = \sum_{i=1}^{n} \sum_{i=1}^{n} w_i w_j \sigma_{ij}$$
- The opportunity set with N risky assets has the same shape as it did with two assets.
- The only difference is that with many assets, some portfolios will fall in the interior of the opportunity set.



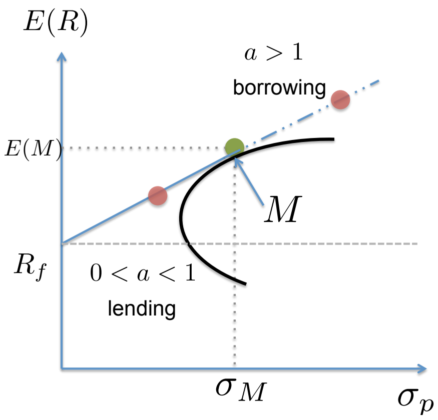
- Risk-free asset, R_f, has zero variance.
- People can borrow and lend at risk-free rate, R_f
- Portfolios along the line drwan from the risk-free rate to any feasible portfolio are possible
- Only one line dominates : the line which is tangent to the efficient frontier

Optimal Portfolio Choice: N Assets and One Risk-Free Asset (Cont 1)



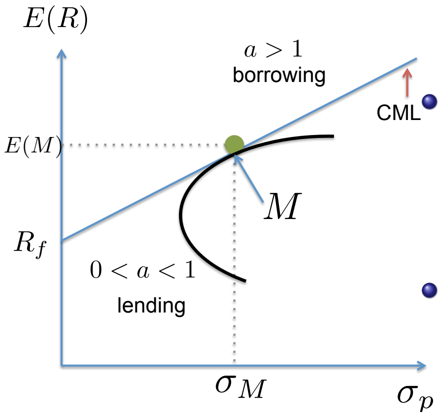
- All investors (regardless of their degree of risk aversion) will prefer combinations of the risk-free asset and portfolio M
- No investor will choose to invest in any other risky portfolio except portfolio M

Optimal Portfolio Choice: N Assets and One Risk-Free Asset (Cont 2)



- ((James Tobin) All investors will hold the same asset portfolio. There are two steps in the investment selection process.
  1. choose the risky portfolio, determined by the line drawn from the risk-free rate of return tangent to the efficient frontier
  2. blend the risky portfolio by borrowing and lending depending on whether we want more risk or less risk

Optimal Portfolio Choice: N Assets and One Risk-Free Asset (Cont 3)



- If investors have homogenous believe, they all have the same linear efficient set; Capital market line (CML)
- The equation for Capital Market Line is