Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

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Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1 - 1.5B + 0.9B^2)R_t = 0.25 + \varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean (μ) = 0 and variance (σ^2) =0.25
B lag-operator

$\boxed{\varepsilon_t \sim (0, 0.25)}$

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

$(1-1.5B+0.9B^2)R_t = 0.25$

$(B^2 - 1.5B + 0.9)R_t = 0.25$

$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\lambda = \frac{-(-1.5) \pm \sqrt{(-1.5)^2-(4)(1)(0.9)}}{2(1)}$

$\lambda_1 = \frac{1.5 + \sqrt{-1.35}}{2}$

$\lambda_2 = \frac{1.5 - \sqrt{-1.35}}{2}$

$\lambda = a + bi$

as $R = \sqrt{a^2+b^2} < 1$

AR(2) model is weak stationarity.

Question 1.2 (10 points)

Your score......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

$(1-1.5B+0.9B^2)R_t = 0.25 + \varepsilon_t$

$R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25 + \varepsilon_t$

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

unconditional mean:

$E[R_t] = E[0.25] + 1.5E[R_{t-1}] - 0.9E[R_{t-2}] + E[\varepsilon_t]$

$= 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + 0$

as weak-stationarity

$E[r_t] = E[r_{t-1}] = E[r_{t-2}] = \dots$

$E[R_t](1-1.5+0.9) = 0.25$

$E[R_t] = \frac{0.25}{(1-1.5+0.9)}$

$= \frac{0.25}{0.4}$

$E[R_t] = 0.625$

conditional mean

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

$E[R_t|\cdot] = E[0.25|\cdot] + 1.5E[R_{t-1}|\cdot] - 0.9E[R_{t-2}|\cdot] + E[\varepsilon_t|\cdot]$

$E[R_t|\cdot] = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + 0$

Question 1.3 (10 points) | Your score......................

unconditional variance

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

$(R_t-\mu) = 1.5(R_{t-1}-\mu) - 0.9(R_{t-2}-\mu) + \varepsilon_t$

$[(R_t-\mu)]^2 = 1.5^2[(R_{t-1}-\mu)]^2 - 0.9^2[(R_{t-2}-\mu)]^2 + \varepsilon_t^2 + 2(1.5)(-0.9)(R_{t-1}-\mu)(R_{t-2}-\mu) + 2(1.5)(1)(R_{t-1}-\mu)(\varepsilon_t) + 2(-0.9)(1)(R_{t-2}-\mu)(\varepsilon_t)$

$E[(R_t-\mu)]^2 = 1.5^2E[(R_{t-1}-\mu)]^2 - 0.9^2[(R_{t-2}-\mu)]^2 + E[\varepsilon_t]^2 + 2(1.5)(-0.9)E[(R_{t-1}-\mu)(R_{t-2}-\mu)] + \cancel{2(1.5)(1)E[(R_{t-1}-\mu)(\varepsilon_t)]} + \cancel{2(-0.9)(1)E[(R_{t-2}-\mu)(\varepsilon_t)]}$

$var(R_t) = 2.25\,var(R_{t-1}) - 0.81\,var(R_{t-2}) \quad -2.7\gamma_1 + 0.25$

as weak-stationarity ; $var(R_t) = var(R_{t-1}) = var(R_{t-2}) \ldots$

$var(R_t) - 2.25\,var(R_t) + 0.81\,var(R_t) = -2.7\gamma_1 + 0.25$

$(1-2.25-0.81)\,var(R_t) = -2.7\gamma_1$

$var(R_t) = \dfrac{-2.7\gamma_1}{-2.06} = 1.31\gamma_1 \dfrac{+0.25}{-2.06}$

$= +1.31\gamma_1 - 0.124$

Conditional variance

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

$var(R_t|\cdot) = var(0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t|\cdot)$

$= var(0.25|\cdot) + 1.5^2\cancel{var(R_{t-1}|\cdot)} - 0.9^2\cancel{var(R_{t-2}|\cdot)} + var(\varepsilon_t|\cdot)$

$= 0 + 0 + 0 + \sigma^2$ $\quad \boxed{\text{Note that } \sigma^2 = 0.25}$

$var(R_t|\cdot) = 0.25$

Question 1.4 (10 points) Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

$(r_t - \mu) = \phi_1(r_{t-1} - \mu) + \phi_2(r_{t-2} - \mu) + \epsilon_t$

$(r_t - \mu) = 1.5(r_{t-1} - \mu) - 0.9(r_{t-2} - \mu) + \epsilon_t$

$(r_t - \mu)(r_{t-l} - \mu) = 1.5(r_{t-1} - \mu)(r_{t-l} - \mu) - 0.9(r_{t-2} - \mu)(r_{t-l} - \mu) + \epsilon_t(r_{t-l} - \mu)$

$E[(r_t - \mu)(r_{t-l} - \mu)] = 1.5E[(r_{t-1} - \mu)(r_{t-l} - \mu)] - 0.9[(r_{t-2} - \mu)(r_{t-l} - \mu)] + E[\epsilon_t(r_{t-l} - \mu)]$

$\gamma_l = 1.5\gamma_{l-1} - 0.9\gamma_{l-2} + 0$

Hence as $\rho_l = \frac{\gamma_l}{\gamma_0}$

$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{(1.5\gamma_0 - 0.9\gamma_{-1})}{\gamma_0} = 1.5 - 0.9\frac{\gamma_{-1}}{\gamma_0} = 1.5 - 0.9\rho_{-1}$

$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{(1.5\gamma_1 - 0.9\gamma_0)}{\gamma_0}$.

Question 1.5 (10 points) Your score.....................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

1step ahead forecast

$R_{t+1} = \phi_0 + \phi_1 R_t + \phi_2 R_{t-1} + \dots + \phi_p R_{t+1-p} + \varepsilon_{t+1}$ — ①

$E[R_{t+1}|\cdot] = \hat{R}_t(1) = \phi_0 + \phi_1 R_t + \phi_2 R_{t-1} + \dots + \phi_p R_{t+1-p} + \cancel{E[\varepsilon_{t+1}|\cdot]}$ — ②

forecasting error ; $R_{t+1} - \hat{R}_t(1) = \varepsilon_{t+1}$

variance of forecasting error ; $var(\hat{R}_t(1)) = var(\varepsilon_{t+1}|\cdot) = \sigma_t^2$

Interval forecasting ; $\hat{R}_t(1) \pm 1.96\sigma_t$

$0.247 \pm 1.96\sigma_t$

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

$R_{t+1} = 0.25 + 1.5R_t - 0.9R_{t-1} + \varepsilon_{t+1}$

$E[R_{t+1}|\cdot] = 0.25 + 1.5(0.01) - 0.9(0.02) + \cancel{E[\varepsilon_{1001}|\cdot]}$

$\hat{R}_t(1) = 0.247$

2-step ahead forecast

$R_{t+2} = \phi_0 + \phi_1 R_{t+1} + \phi_2 R_t + \phi_3 R_{t-1} + \dots + \phi_p R_{t+2-p} + \varepsilon_{t+2}$ — ①

$E[R_{t+2}|\cdot] = \hat{R}_t(2) = \phi_0 + \phi_1 E[R_{t+1}|\cdot] + \phi_2 R_t + \dots + \phi_p R_{t+2-p} + E[\varepsilon_t|\cdot]$ — ②

Forecasting error ; $R_{t+2} - \hat{R}_t(2) = \phi_1(R_{t+1} - \hat{R}_t(1)) + \varepsilon_{t+2}$

Variance of forecasting error ; $var(\hat{R}_t(2)) = var[\phi_1(R_{t+1} - \hat{R}_t(1)) + \varepsilon_{t+2}|\cdot]$

$= var[\phi_1(\varepsilon_{t+1})|\cdot] + var(\varepsilon_{t+2}|\cdot)$

$= \phi_1^2\sigma_t^2 + \sigma_t^2$

$= (1+\phi_1^2)\sigma_t^2$

$\hat{R}_t(2) = 0.25 + 1.5E[R_{t+1}|\cdot] - 0.9R_t + 0$

$= 0.25 + 1.5(0.247) - 0.9(0.01)$

$= 0.6115$

Interval forecasting ; $\hat{R}_t(2) \pm 1.96\sqrt{(1+\phi_1^2)\sigma_t^2}$

$0.6115 \pm 1.96\sqrt{(1+\phi_1^2)\sigma_t^2}$

∞-step ahead forecasting

. forecast value when $l \to \infty$ will approach $E(r_t)$ at the long run.

$r_t \pm 1.96\sqrt{var(r_t)}$