

Non-linearities and Time-Varying Volatility Models

Autoregressive Conditional Heteroscedasticity

ARCH(1) Model

ARCH(q) Model

ARCH in Mean (ARCH-M) Model

Generalized ARCH or GARCH(p,q) Model

Asymmetric GARCH Models

- GJR or Threshold GARCH (TARCH) Model
- Exponential GARCH (EGARCH) Model

ARCH(I) Model

$$y_t = \beta' x_t + \varepsilon_t$$

$$\varepsilon_t = u_t \sqrt{\alpha_0 + \alpha_1 \varepsilon_{t-1}^2}$$

where $E[\varepsilon_t | x_t, \varepsilon_{t-1}] = 0$ then $E[\varepsilon_t | x_t] = 0$

Classical regression model follows OLS assumptions (homoscedasticity).

However, the conditional variance is:

$$\begin{aligned} \text{Var}[\varepsilon_t | \varepsilon_{t-1}] &= E[\varepsilon_t^2 | \varepsilon_{t-1}] = E[u_t^2 [\alpha_0 + \alpha_1 \varepsilon_{t-1}^2]] \\ &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \end{aligned}$$

ARCH(1) Model

Since classical regression model follows OLS assumptions, OLS estimators are efficient.

However, with the conditional heteroscedasticity, nonlinear estimation method (such as MLE) can provide more efficient estimators.

Conditional log-likelihood of the model:

$$\log L = -\frac{T}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^T \log(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2) - \frac{1}{2} \sum_{t=1}^T \frac{\varepsilon_t^2}{\alpha_0 + \alpha_1 \varepsilon_{t-1}^2}$$

where $\varepsilon_t = y_t - \beta' x_t$

ARCH(q) Model

$$y_t = \beta'x_t + \varepsilon_t$$

ARCH(q) process:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$$

ARCH-in-Mean Model

$$y_t = \beta'x_t + \delta \sigma_t^2 + \varepsilon_t$$

$$\text{Var}[\varepsilon_t | \Psi_t] = \text{ARCH}(m)$$

Test for ARCH Effects

Hypothesis: $H_0: \alpha_1 = \alpha_2 = \dots = \alpha_q = 0$

1. Estimate $y_t = \beta'x_t + \varepsilon_t$ obtain $\hat{\varepsilon}_t$

2. Estimate

$$\hat{\varepsilon}_t^2 = \alpha_0 + \alpha_1 \hat{\varepsilon}_{t-1}^2 + \alpha_2 \hat{\varepsilon}_{t-2}^2 + \dots + \alpha_q \hat{\varepsilon}_{t-q}^2 + u_t$$

obtain R^2

3. Calculate $TR^2 \sim \chi^2(q)$

GARCH(p,q) Model

$$y_t = \beta'x_t + \varepsilon_t$$

GARCH(p,q) process:

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + \delta_1\sigma_{t-1}^2 + \delta_2\sigma_{t-2}^2 + \cdots + \delta_p\sigma_{t-p}^2 \\ &\quad + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \cdots + \alpha_q\varepsilon_{t-q}^2 \\ &= \alpha_0 + \sum_{j=1}^p \delta_j\sigma_{t-j}^2 + \sum_{i=1}^q \alpha_i\varepsilon_{t-i}^2\end{aligned}$$

Asymmetric GARCH Model

Glosten, Jagannathan & Runkle (GJR) or
Threshold GARCH (TARCH) Model

$$\text{TARCH}(p,q,r) \quad y_t = \beta'x_t + \varepsilon_t$$

Conditional variance process:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^p \delta_j \sigma_{t-j}^2 + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2 + \sum_{k=1}^r \gamma_k \varepsilon_{t-k}^2 I_{t-k}$$

where $I_{t-1} = 1$ if $\varepsilon_{t-1} < 0$
= 0 otherwise

Asymmetric GARCH Model

Exponential GARCH (EGARCH) Model

$$\text{EGARCH}(p,q,r) \quad y_t = \beta'x_t + \varepsilon_t$$

Conditional variance process:

$$\ln(\sigma_t^2) = \alpha_0 + \sum_{j=1}^p \delta_j \ln(\sigma_{t-j}^2) + \sum_{i=1}^q \alpha_i \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| + \sum_{k=1}^r \gamma_k \frac{\varepsilon_{t-k}}{\sigma_{t-k}}$$