The Logic of Quantified Statements

TU152: Fundamental Mathematics

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Predicates

Definition

- A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
- The **domain of a predicate variable** is the set of all values that may be substituted in place of the variable in the *predicate*.

Example: Let $Q(x, y) : x = y + 3$ with domain the collection of numbers $0, 1, 2, 3 \ldots$.
What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?
Answer:

Truth Set of a Predicate

Example: Let $P(x)$ be the predicate "$x^3 > x$" with domain the set $\mathbb{R}$ of all real numbers. Write

- $P(1)$ and $P(-1)$
- $P(\frac{1}{3})$
- $P(3)$

and indicate which of these statements are true and which are false.

Truth Set of a Predicate

Definition

- If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.

  The **truth set** of $P(x)$ is denoted

$$\{x \in D | P(x)\}$$

  which is read "the set of all x in D such that $P(x)$."

Example Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if

(i) the domain of n is the set $\mathbb{Z}^+$ of all positive integers
(ii) the domain of n is the set $\mathbb{Z}$ of all integers.

Answer:

Notation: $\mathbb{R}, \mathbb{Z}, \mathbb{N}$

Notations

- $\mathbb{R}$ is the set of all real numbers.
- $\mathbb{R}^+$ is the set of all positive real numbers.
- $\mathbb{R}^-$ is the set of all negative real numbers.
- $\mathbb{Z}$ is the set of all integers.
- $\mathbb{Z}^+$ is the set of all integers.
- $\mathbb{Z}^-$ is the set of all integers.
- $\mathbb{Q}$ is the set of all rational numbers. I.e. $\mathbb{Q} = \{\frac{a}{b} | a \in \mathbb{Z}, b \in \mathbb{Z} - \{0\}\}$
- $\mathbb{N}$ is the set of *natural* numbers. In general, $\mathbb{N}$ is just the set of all positive integers. However, it is possible to have 0 in some text books.

Multiple Quantifiers Arguments

Universal Quantifier: $\forall$

Another way to generate propositions is by means of quantifiers. Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true

Definition

Let $Q(x)$ be a predicate and D the domain of x. A universal statement is a statement of the form

$$\forall x \in D, Q(x),$$

where the symbol $\forall$ denotes "for all" or "for any."

- It is defined to be **true** if, and only if, $Q(x)$ is true for **every** x in D.
- It is defined to be **false** if, and only if, $Q(x)$ is false for **at least one** x in D.
- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Multiple Quantifiers Arguments

Universal Quantifier: $\forall$

Example:

(i) Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, \quad x^2 \geq x.$$

Show that this statement is true.

(ii) Consider the statement

$$\forall x \in \mathbb{R}, \qquad x^2 \geq x.$$

Find a counterexample to show that this statement is false.

Answer:

Multiple Quantifiers Arguments

The Existential Quantifier: $\exists$

Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form

$$\exists x \in D \quad \text{such that} \quad Q(x).$$

where the symbol $\exists$ denotes "there exists."

- It is defined to be **true** if, and only if, $Q(x)$ is **true** *for at least one* x in D.
- It is **false** if, and only if, $Q(x)$ is **false** for all x in D.

Example:

Consider the statement

$$\exists x \in \mathbb{Z}^+ \quad \text{such that} \quad x^2 = x.$$

Show that this statement is true.

Answer:

The Existential Quantifier: $\exists$

Example: Let $D = \{5, 6, 7, 8\}$. Determine the truth value of the following statements:

$$\exists x \in D \quad \text{such that} \quad x^2 = x.$$

Answer:

Example: Let $P(x)$ denote the statement $x > 3$: Determine the truth value of the statement:

$$\exists x \in \mathbb{R}, P(x).$$

Answer:

The Existential Quantifier: $\exists$

Example: Determine the truth value of:
(i) $\forall x \in \mathbb{R}, x > \frac{1}{x}$
(ii) $\exists x \in \mathbb{R}, x > \frac{1}{x}$.

Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the universal conditional statement:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $\boxed{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently,
$$\forall x, P(x) \rightarrow Q(x).$$
- The notation $\boxed{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently,
$$\forall x, P(x) \leftrightarrow Q(x).$$

Universal Conditional Statements

Example: Let the domain of x be the set of all positive integers $\mathbb{Z}^+$.
Consider the two predicates $P(x)$: x is a factor of 4 and Q(x) : x is a factor of 8.
Show that $P(x) \Rightarrow Q(x)$.

Negation of a Universal Statement

Theorem (Negation of a Universal Statement)

The negation of a statement of the form

$$\forall x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D, \sim Q(x).$$

Symbolically,

$$\sim (\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x).$$

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Example:
Suppose the statement

All mathematicians wear glasses

is false. So a correct negation is its negation :

There is at least one mathematician who does not wear glasses.

Negation of an Existential Statement

Theorem (Negation of an Existential Statement)

The negation of a statement of the form

$$\exists x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim Q(x).$$

Symbolically,

$$\sim (\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x).$$

The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Example: Rewrite the following statement formally. Then write formal and informal negations.

No politicians are honest.

Answer:

- Formal version:
- Formal negation:
- Informal negation:

Negation of Universal & Existential Statements

Example: Write formal negations for the following statements:

- $\forall$ primes p, p is odd.
- $\exists$ a triangle T such that the sum of the angles of T equals 200°.

Answer

Negation of a Universal Conditional Statement

Negation of a Universal Conditional Statement

$$\sim (\forall x, P(x) \to Q(x)) \equiv \exists x, P(x) \wedge \sim Q(x)$$

Example Write the negation of each of the following statements:

i $\forall x \in \mathbb{R}; x > 3 \to x^2 > 9$
ii Every polynomial function is continuous.
iii There exists a triangle with the property that the sum of angles is greater than 180°:

Answer:

Contrapositive, Converse, and Inverse of a Universal Conditional Statement

Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

- Its contrapositive is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
- Its converse is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
- Its inverse is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example:
Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Necessary and Sufficient Conditions, Only If

The definitions of necessary, sufficient, and only if can also be extended to apply to universal conditional statements.

Definition

- $\forall x, r(x)$ is a sufficient condition for $s(x)$, means $\forall x, r(x) \to s(x)$
- $\forall x, r(x)$ is a necessary condition for s(x) means $\forall x, \sim r(x) \to \sim s(x)$ or $\forall x, s(x) \to r(x)$.
- $\forall x, r(x)$ only if $s(x)$ means $\forall x, \sim s(x) \to \sim r(x)$ or, equivalently, $\forall x, r(x) \to s(x)$.

Multiple Quantifiers

Interpreting Statements with Two Different Quantifiers

- To determine the truth of a statement of the form

$$\boxed{\forall x \in D\ \exists y \in E, P(x,y)} \text{ or}$$

$$\forall x \in D, \exists y \in E \text{ such that } P(x,y),$$

we have to show that:
for whatever element x in D is chosen, we must find an element y in E that "works" (i.e. $P(x,y)$ is true) for that particular x.

- To determine the truth of a statement of the form

$$\boxed{\exists x \in D\ \forall y \in E,\ P(x,y)} \text{ or}$$

$$\exists x \in D \text{ such that } \forall y \in E,\ P(x,y),$$

we have to *find one particular x in D that will "work" (i.e. $P(x,y)$ is true) for every element y in E.*

Multiple Quantifiers

Example:

(i) Let $P(x,y)$ denote the statement "$x + y = y + x$." Determine the truth value of the statement

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\quad P(x,y).$$

(ii) Let $Q(x,y)$ denote the statement "$x + y = 0$." Determine the truth value of the statement

$$\exists y \in \mathbb{R}\ \forall x \in \mathbb{R},\quad Q(x,y)$$

Answer:

Multiple Quantifiers

Example: Let $P(x, y)$ denote the statement "$xy = 1$," where the domain of x is the set of positive integers $\mathbb{Z}^+$ and the domain of y is the set of all real numbers $\mathbb{R}$.
Determine the truth value of the following statements.

(i) For every positive integer x and for every real number y, "$xy = 1$. "

$$\forall x \in \mathbb{Z}^+ \ \forall y \in \mathbb{R}, P(x, y)$$

(ii) For every positive integer x there is a real number y such that $xy = 1$.

$$\forall x \in \mathbb{Z}^+ \ \exists y \in \mathbb{R}, P(x, y)$$

(iii) There exists a real number y such that, for every positive integer x, $xy = 1$.

$$\exists y \in \mathbb{R} \ \forall x \in \mathbb{Z}^+, P(x, y)$$

Answer:

Order of Quantifiers

Answer: (Continued)

Negations of Multiply-Quantified Statements

Recall that, if we let $Q(x)$ be a predicate and D be the domain of x

$$\sim\Big(\forall x, Q(x)\Big) \equiv \exists x, \sim Q(x)$$

and

$$\sim\Big(\exists x, Q(x)\Big) \equiv \forall x, \sim Q(x).$$

Negations of Multiply-Quantified Statements

- $$\sim\Big(\forall x \in D,\ \exists y \in E, P(x, y)\Big) \equiv \exists x \in D, \forall y \in E, \sim P(x, y)$$
- $$\sim\Big(\exists x \in D,\ \forall y \in E, P(x, y)\Big) \equiv \forall x \in D, \exists y \in E, \sim P(x, y)$$

In general, if we explicitly define D to be the domain of x and E to be the domain of y, then we can also write:

$$\sim\Big(\forall x \exists y, P(x, y)\Big) \equiv \exists x \forall y, \sim P(x, y)$$

and

$$\sim\Big(\exists x \forall y, P(x, y)\Big) \equiv \forall x \exists y, \sim P(x, y)$$

Negations of Multiply-Quantified Statements

Show that

$$\sim\Big(\forall x \exists y, P(x, y)\Big) \equiv \exists x \forall y, \sim P(x, y).$$

From $\sim\Big(\forall x, Q(x)\Big) \equiv \exists x, \sim Q(x)$, and $\sim\Big(\exists x, Q(x)\Big) \equiv \forall x, \sim Q(x)$

$$\begin{aligned} \sim\Big(\forall x \exists y, P(x, y)\Big) &\equiv \sim\Big(\forall x(\exists y, P(x, y))\Big) \\ &\equiv \exists x, \sim\Big(\exists y, P(x, y)\Big) \\ &\equiv \exists x \forall y, \sim P(x, y) \end{aligned}$$

■

Show that

$$\sim\Big(\exists x \forall y, P(x, y)\Big) \equiv \forall x \exists y, \sim P(x, y).$$

From $\sim\Big(\forall x, Q(x)\Big) \equiv \exists x, \sim Q(x)$, and $\sim\Big(\exists x, Q(x)\Big) \equiv \forall x, \sim Q(x)$

$$\begin{aligned} \sim\Big(\exists x \forall y, P(x, y)\Big) &\equiv \sim\Big(\exists x(\forall y, P(x, y))\Big) \\ &\equiv \end{aligned}$$

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Negations of Multiply-Quantified Statements

Example: Let D_x, D_y, and D_z be the domains for x, y, and z, respectively. Express the negations of the statement:

$$\forall x \exists y \forall z, T(x, y, z)$$

so that all negation symbols $\sim$ precede predicates.

Answer:

$$\sim \left(\forall x \exists y \forall z, T(x, y, z)\right) \equiv \sim \forall x \left(\exists y \forall z, T(x, y, z)\right)$$
$$\equiv$$

Order of Quantifiers

Example: Let $R(x, y)$ be the predicate "x understands y," where the domain of x is the set of students in this TU152 class and the domain of y is the set of examples in these lecture notes. Write the following statements using the quantifiers $\forall$, $\exists$, and the predicate $R(x, y)$.

(1) There exists a student in this class who understands every example in these lecture notes.
Answer:

(2) For every example in these lecture notes there is at least one student in the class who understands that particular example (it is possible that different students understand different examples).
Answer:

(3) Every student in this class understands at least one example in these notes.
Answer:

(4) There is an example in these notes that every student in this class understands.
Answer:

Notice

$$\exists x \forall y, R(x, y) \not\equiv \forall y \exists x, R(x, y) \quad \text{and} \quad \forall x \exists y, R(x, y) \not\equiv \exists y \forall x, R(x, y)$$

Negations of Multiply-Quantified Statements

Example: Let D_x, D_y be the domains for x, y, respectively. Express the negations of the statement:

$$\forall x \exists y, P(x, y) \vee \forall x \exists y, Q(x, y)$$

so that all negation symbols $\sim$ precede predicates.

Answer:

$$\sim \left(\forall x \exists y, P(x, y) \vee \forall x \exists y, Q(x, y)\right) \equiv$$
$$\equiv$$

Arguments with Quantified Statements

The rule of universal instantiation

"If some property is true of everything in a set, then it is true of any particular thing in the set."

Example:

All human beings are mortal.
John is a human being.
∴ John is mortal.

- Universal instantiation is the fundamental tool of deductive reasoning.
- Mathematical formulas, definitions, and theorems are like general templates that are used over and over in a wide variety of particular situations.
- A given theorem says that such and such is true for all things of a certain type. If, in a given situation, you have a particular object of that type, then by universal instantiation, you conclude that such and such is true for that particular object.

Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called universal modus ponens.

Universal Modus Ponens

Formal Version	**Informal Version**
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore Q(a)$.	$\therefore$ a makes $Q(x)$ true.

Universal Modus Ponens

Example: Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.
k is a particular integer that is even.
$\therefore$ k^2 is even.

Answer:

Universal Modus Tollens

Universal modus tollens is the main concept of proof by contradiction, which is one of the most important methods of mathematical argument.

Universal Modus Tollens

Formal Version	**Informal Version**
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$ for a particular a.	a does not make $Q(x)$ true.
$\therefore \sim P(a)$.	$\therefore$ a does not make $P(x)$ true.

Universal Modus Tollens

Example: Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All lawyers went to law schools.
Tom didn't go to a law school.
$\therefore$ Tom is not a lawyer.

Answer:

Universal Transitivity

Universal Transitivity

Formal Version	Informal Version
$\forall x, P(x) \rightarrow Q(x).$	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\forall x, Q(x) \rightarrow R(x).$	If x makes $Q(x)$ true, then x makes $R(x)$ true.
$\therefore \forall x, P(x) \rightarrow R(x).$	$\therefore$ If x makes $P(x)$ true, then x makes $R(x)$ true.

Validity of Arguments with Quantified Statements

Definition

- To say that an argument form is **valid** means the following:

  No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.
- An argument is called **valid** if, and only if, its form is valid.

Using Diagrams to Show Validity& Invalidity

Example: Use a diagram to show the validity of the following argument:

All lawyers went to law schools.
Tom didn't go to a law school.
$\therefore$ Tom is not a lawyer.

Using Diagrams to Show Validity& Invalidity

Example: Use a diagram to show the invalidity of the following argument:

All lawyers went to law schools.
Tom went to a law school.
$\therefore$ Tom is a lawyer.

Using Diagrams to Show Validity& Invalidity

Example: Use a diagram to show the invalidity of the following argument:

All freshmen must take TU152.

Jane takes TU152.

∴ Jane is a freshman.

Converse & Inverse Errors (Quantified Form)

The following arguments are invalid.

Converse Error (Quantified Form)

Formal Version	**Informal Version**
$\forall x, P(x) \rightarrow Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore P(a)$.	$\therefore$ a makes $P(x)$ true.

Inverse Error (Quantified Form)

Formal Version	**Informal Version**
$\forall x, P(x) \rightarrow Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$ for a particular a.	a does not make $P(x)$ true.
$\therefore \sim Q(a)$.	$\therefore$ a does not make $Q(x)$ true.

Exercises: Rewrite the following arguments using quantifiers, variables, and predicate symbols. Determine if these arguments are valid. Explain your answers.

(a)

All human beings are mortal.

Buster is mortal.

∴ **Buster** is a human being.

(b)

Any sum of two rational numbers is rational.

The sum $r + s$ is rational.

∴ The numbers r and s are both rational.

(c)

All freshmen must take TU152.

Jane is a freshman.

∴ Jane must take TU152.

(d)

All healthy people eat an apple a day

Jane eats an apple a day.

∴ Jane is a healthy person.