Maximum Likelihood Estimation Method

The CAPM model can be stated as:

$$y_t = \alpha + \beta x_t + u_t \quad (1)$$

where: y_t = Excess return of stock at time t.

x_t = Market excess return at time t.

u_t = Disturbance term at time t.

```
. reg y x

      Source |       SS           df       MS      Number of obs   =       240
-------------+----------------------------------   F(1, 238)       =    565.24
       Model |  4691.00669         1  4691.00669   Prob > F        =    0.0000
    Residual |  1975.18377       238  8.29909148   R-squared       =    0.7037
-------------+----------------------------------   Adj R-squared   =    0.7025
       Total |  6666.19046       239  27.8920103   Root MSE        =    2.8808

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .9315372   .0391816    23.77   0.000     .8543501    1.008724
       _cons |   .1978772   .1886373     1.05   0.295    -.1737347    .5694892
------------------------------------------------------------------------------
```

1. Cauchy Distribution

If the disturbances term u_t are assumed to be Identical Independent Distribution (IID) with Cauchy distribution with density:

$$f(u_t) = (\pi(1+u_t^2))^{-1}$$

To estimate the model using maximum likelihood, we first create program file:

```
program define ml_cauchy
        args lnf Xb
        tempvar res
        quietly gen double `res' = $ML_y1 - `Xb'
        replace `lnf' = ln(1/(_pi*(1+(`res')^2)))
end
```

```
. ml model lf ml_cauchy (y=x)

. ml maximize

(240 real changes made)
initial:       log likelihood = -847.98921
(240 real changes made)
(240 real changes made)
alternative:   log likelihood =  -834.0182
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
rescale:       log likelihood = -821.66632
(240 real changes made)
(240 real changes made)
```

```
(240 real changes made)
(240 real changes made)
Iteration 0:   log likelihood = -821.66632  (not concave)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
Iteration 1:   log likelihood = -625.87317
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
Iteration 2:   log likelihood = -617.63957
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
Iteration 3:   log likelihood = -617.44101
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
(240 real changes made)
Iteration 4:   log likelihood = -617.44095

                                                  Number of obs   =        240
                                                  Wald chi2(1)    =    1455.04
Log likelihood = -617.44095                       Prob > chi2     =     0.0000

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .9145708   .0239762    38.14   0.000     .8675783    .9615632
       _cons |   .1363996   .1215726     1.12   0.262    -.1018784    .3746776
------------------------------------------------------------------------------
```

The default algorithm in estimating ML is Newton-Raphson (nr), if we would like to use BHHH algorithm, we can use the following command.

```
. ml model lf ml_cauchy (y=x), tech(bhhh)

. ml maximize

(240 real changes made)
initial:       log likelihood = -847.98921
...
Iteration 5:   log likelihood = -617.44095

                                                  Number of obs   =        240
                                                  Wald chi2(1)    =    1969.66
Log likelihood = -617.44095                       Prob > chi2     =     0.0000

------------------------------------------------------------------------------
             |                 OPG
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |    .914568   .0206073    44.38   0.000     .8741785    .9549575
       _cons |   .1364103   .0914454     1.49   0.136    -.0428195      .31564
------------------------------------------------------------------------------
```

2. Normal Distribution

If the disturbances term u_t are assumed to be normal distribution. To estimate the model using maximum likelihood, we use the following command:

```
program ml_norm
     version 10
     args todo b lnf
     tempvar theta1 sigma
     mleval `theta1' = `b', eq(1)
     mleval `sigma' = `b', eq(2)
     tempvar res
     quietly gen double `res' = $ML_y1-`theta1'
     mlsum `lnf' = -0.5*ln(2*_pi)-ln(`sigma')-0.5*((`res'/ `sigma')^ 2)
end
```

```
. ml model d0 ml_norm (y=x) ()

. ml maximize

initial:       log likelihood =     -<inf>  (could not be evaluated)
feasible:      log likelihood = -13484.369
rescale:       log likelihood = -831.27985
rescale eq:    log likelihood = -761.59207
Iteration 0:   log likelihood = -761.59207  (not concave)
Iteration 1:   log likelihood = -606.96261
Iteration 2:   log likelihood = -593.61441
Iteration 3:   log likelihood = -593.47896
Iteration 4:   log likelihood = -593.47858
Iteration 5:   log likelihood = -593.47858

                                                Number of obs   =        240
                                                Wald chi2(1)    =     569.99
Log likelihood = -593.47858                     Prob > chi2     =     0.0000

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
eq1          |
           x |   .9315372    .039018    23.87   0.000     .8550633    1.008011
       _cons |   .1978772   .1878496     1.05   0.292    -.1703013    .5660558
-------------+----------------------------------------------------------------
eq2          |
       _cons |   2.868786   .1309416    21.91   0.000     2.612145    3.125427
------------------------------------------------------------------------------

. test x=0

 ( 1)  [eq1]x = 0

           chi2(  1) =  569.99
         Prob > chi2 =    0.0000

. est store unres
```

In case of assuming normal distribution, the estimated results are the same as estimating the model using OLS, except the estimated result of standard error of regression (standard deviation of the error term (σ)). In OLS, Root MSE = 2.8808 while in MLE, sigma _cons = 2.868786

```
. regress y x

      Source |       SS       df       MS              Number of obs =     240
-------------+------------------------------           F(  1,   238) =  565.24
       Model |  4691.00669     1  4691.00669           Prob > F      =  0.0000
```

```
    Residual |  1975.18377   238  8.29909148           R-squared     =  0.7037
-------------+------------------------------           Adj R-squared =  0.7025
       Total |  6666.19046   239  27.8920103           Root MSE      =  2.8808

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .9315372   .0391816    23.77   0.000     .8543501    1.008724
       _cons |   .1978772   .1886373     1.05   0.295    -.1737347    .5694892
------------------------------------------------------------------------------

. ml model d0 ml_norm (y=) ()

. ml maximize

initial:       log likelihood =     -<inf>  (could not be evaluated)
feasible:      log likelihood = -13484.369
rescale:       log likelihood = -831.27985
rescale eq:    log likelihood = -761.59207
Iteration 0:   log likelihood = -761.59207
Iteration 1:   log likelihood = -739.46007
Iteration 2:   log likelihood = -739.44504
Iteration 3:   log likelihood = -739.44504

                                                  Number of obs   =        240
                                                  Wald chi2(0)    =          .
Log likelihood = -739.44504                       Prob > chi2     =          .

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
eq1          |
       _cons |   .9513828   .3401948     2.80   0.005     .2846133    1.618152
-------------+----------------------------------------------------------------
eq2          |
       _cons |   5.270275    .240554    21.91   0.000     4.798798    5.741752
------------------------------------------------------------------------------

. est store res

. lrtest unres res

Likelihood-ratio test                                  LR chi2(1)  =    291.93
(Assumption: res nested in unres)                      Prob > chi2 =    0.0000
```