The Logic of Compound Statements II

1 Valid and Invalid Arguments: Introduction & Definitions

Definition 1.1. An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms.

- *All statements in an argument and all statement forms in an argument form, except for the final one*, are called **premises** (or **assumptions** or **hypotheses**).
- The final statement or statement form is called the **conclusion**.
- The symbol "$\therefore$" , which is called *"therefore"* is normally placed just *before the conclusion.*

Valid & Invalid Arguments

- To say that an argument form is **valid** means that...

  whenever the premises are all true, then the conclusion is also true,

  no matter what particular statements are substituted for the statement variables in its premises.

- An argument is **invalid** means that there is an argument of that form whose premises are all true and whose conclusion is false.

When an argument is valid and its premises are true, the truth of the conclusion is said to be *inferred* or *deduced* from the truth of the premises.

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. Find rows in which all premises are true. Each of these row is called a **critical row**.
   - If there is a critical row in which the conclusion is false, then *the argument form is invalid.*
   - If the conclusion in every critical row is true, then *the argument form is valid.*

Example 1.1. Show that the statements:

"If Mark is a lawyer,then he went to a law school. Mark is a lawyer.
Therefore Mark went to a law school."

form an argument. Determine if this argument is valid or invalid.
Solution Let

p: Mark is a lawyer.

r: Mark went to law school.

Then this argument can be written as:

Truth table:

p	r	$p \rightarrow r$	p	r
T	T			
T	F			
F	T			
F	F			

- Recall that the critical row is the row in the truth table which has all premises true.
- In this case, the premises are ....
- Hence, from the table above that, the only critical row is ....
- This argument is **.................**, because ....

■

Example 1.2. Show that the argument

$$\begin{array}{ll} & p \rightarrow q \\ & q \rightarrow p \\ \therefore & p \vee q \end{array}$$

is invalid.
Solution: Truth table for this argument can be written as follows.

- In this case, the premises are ....
- Hence, from the table above that, the critical rows are ....
- This argument is **.................**, because ....

2 Rules of Inference

2.1 Modus Ponens and Modus Tollens

Definition 2.1.
- A **rule of inference** is a form of argument that is valid.
- An argument form consisting of two premises and a conclusion is called a **syllogism**. The first and second premises are called the *major premise* and *minor premise*, respectively.
  - Modus Ponens
  - Modus Tollens

Valid Arguments: Modus Ponens

Modus Ponens (the method of affirming)

$$\begin{array}{ll} & p \to q \\ & p \\ \therefore & q \end{array}$$

Truth table:

p	r	$p \to r$	p	r	
T	T	T	T	T	←-- critical row
T	F	F	T	F	
F	T	T	F	T	
F	F	T	F	F	

This argument is **valid** because all the critical row has true conclusion.

Example 2.1. The follows are **valid** arguments by Modus Ponens.

(a)

If it rains, my car gets wet.
It rains.
∴ ....

(b)

If the sum of the digits of 4,791 is divisible by 3, then 4,791 is divisible by 3.
The sum of the digits of 4,791 is divisible by 3.
∴ 4,791 is divisible by 3.

Modus Tollens (the method of denying)

$$\begin{array}{rl} & p \to q \\ & \sim q \\ \therefore & \sim p \end{array}$$

Truth table:

Example 2.2. The follows are **valid** arguments by Modus Tollens.

(a)

If it rains, my car gets wet.

....

$\therefore$ ....

(b)

If the sum of the digits of 4,790 is divisible by 3, then 4,790 is divisible by 3.
4,790 is not divisible by 3.
$\therefore$ The sum of the digits of 4,790 is not divisible by 3.

Example 2.3. Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

1. If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole. There are more pigeons than there are pigeonholes.

   $\therefore$ ....

2. If 870,232 is divisible by 6, then it is divisible by 3. 870,232 is not divisible by 3.

   $\therefore$ ....

2.2 Other rules of inferences

We will consider different types of valid arguments by other rules of inferences.

Generalization (I) $\begin{array}{ll} & p \\ \therefore & p \vee q \end{array}$ (II) $\begin{array}{ll} & q \\ \therefore & p \vee q \end{array}$

These argument forms are used for making generalizations. For instance, according to the first, if p is true, then, more generally, "p or q" is true for any other statement q. This can be proved by using the truth table.

Truth table:

Example 2.4. Determine if the following arguments are valid or not.

(a) I have a car
Therefore I have a house or a car.

(b) $\begin{array}{ll} & p \\ \therefore & p \vee q \vee r \vee s \end{array}$

Specialization (I) $\begin{array}{ll} & p \wedge q \\ \therefore & p \end{array}$ (II) $\begin{array}{ll} & p \wedge q \\ \therefore & q \end{array}$

These argument forms are used for specializing. When classifying objects according to some property, you often know much more about them than whether they do or do not have that property. When this happens, you discard extraneous information as you concentrate on the particular property of interest.
Truth table:

Example 2.5. The following arguments are valid by the specialization rule.

(a) I have a car and I have a house
Therefore I have a house.

(b) $\begin{array}{ll} & p \wedge q \wedge r \wedge s \\ \therefore & p \end{array}$

Elimination (I) $p \vee q$, $\sim p$, $\therefore\ q$ (II) $p \vee q$, $\sim q$, $\therefore\ p$

$$\text{(I)}\quad \begin{array}{rr} & p \vee q \\ & \sim p \\ \therefore & q \end{array} \qquad\qquad \text{(II)}\quad \begin{array}{rr} & p \vee q \\ & \sim q \\ \therefore & p \end{array}$$

These argument forms say that when you have only two possibilities and you can rule one out, the other must be the case.
Truth table for the elimination rules:

(I)

p	q	$p \vee q$	$\sim p$	q	
T	T	T	F	T	
T	F	T	F	F	
F	T	T	T	T	←-- critical row
F	F	F	T	F	

(II)

p	q	$p \vee q$	$\sim q$	p	
T	T	T	F	T	
T	F	T	T	T	←-- critical row
F	T	T	F	F	
F	F	F	T	F	

Each of the arguments (I) and (II) in the "elimination" inference rule is **valid** because every critical row has true conclusion. ■

Example 2.6. The following arguments are valid by the elimination rule.
(a)
I have a car or I have a house
I do not have a house

Therefore,...
(b) Suppose you solve for a **positive** number x such that $(x-1)(x+1) = 0$.

$$\begin{array}{lll} & x-1=0 \quad \textit{or} & x+1=0 \\ & & x+1 \neq 0 \\ \therefore & & x-1=0 \end{array}$$

Contradiction Rule

- If you can show that the supposition that p is false leads logically to a contradiction, then you can conclude that p is true.
- If an assumption leads to a contradiction, then that assumption must be false.

Let c be a contradiction. Then

$$\begin{array}{rl} & \sim p \to c \\ \therefore & p \end{array}$$

is a valid argument.

Truth table:

Rules of Inferences

Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore\ q$	Elimination	(i) $p \vee q$ $\sim q$ $\therefore\ p$ (ii) $p \vee q$ $\sim p$ $\therefore\ q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore\ \sim p$	Transitivity	$p \to q$ $q \to r$ $\therefore\ p \to r$
Generalization	(i) p $\therefore\ p \vee q$ (ii) q $\therefore\ p \vee q$	Proof by Division into Cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore\ r$
Specialization	(i) $p \wedge q$ $\therefore\ p$ (ii) $p \wedge q$ $\therefore\ q$	Contradiction Rule	$\sim p \to c$ $\therefore\ p$
Conjunction	p q $\therefore\ p \wedge q$		

3 Fallacies

Definition 3.1 (Fallacies)**.** A fallacy is an error in reasoning that results in an invalid argument.

Invalid argument: Converse Error

$$\begin{array}{rr} & p \to q \\ & q \\ \therefore & p \end{array}$$

Invalid argument: Inverse Error

$$\begin{array}{rr} & p \to q \\ & \sim p \\ \therefore & \sim q \end{array}$$

Truth table for Converse Error:

p	q	$p \to q$	q	p	
T	T	T	T	T	←-- critical row
T	F	F	F	T	
F	T	T	T	F	←-- critical row: This has true premises but false conclusion.
F	F	T	F	F	

This argument is **invalid** because there is a critical row with false conclusion.
Similarly, the truth table can be used to show that the argument in the form of **inverse error** is **invalid**. ■

Example 3.1. (Converse Error):

Consider the following argument.

If John cheated on the exam, then John sits in the back row.
John sits in the back row.
∴ John cheated on the exam.

(a) Show that the following argument is invalid.

(b) Construct an inverse error from this argument.

Solution: Let p be " John cheated on the exam" and q be "John sits in the back row."

Example 3.2. (Exercise)
You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

(b) If my glasses are on the kitchen table, then I saw them at breakfast.

(c) I did not see my glasses at breakfast.

(d) I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Solution:
Let
RK = I was reading the newspaper in the kitchen.
GK = My glasses are on the kitchen table.
SB = I saw my glasses at breakfast.
RL = I was reading the newspaper in the living room.
GC = My glasses are on the coffee table.

Here is a sequence of steps you might use to reach the answer, together with the rules of inference that allow you to draw the conclusion of each step:

(1) RK $\rightarrow$ GK by (a)
GK $\rightarrow$ SB by (b)

$\therefore$ RK $\rightarrow$ SB by transitivity

(2) RK $\rightarrow$ SB by the conclusion of (1)
$\sim$ SB by (c)

$\therefore$ $\sim$ RK by modus tollens

(3) RL $\vee$ RK by (d)
$\sim$ RK by the conclusion of (2)

$\therefore$ RL by elimination

(4) RL $\rightarrow$ GC by (e)
RL by the conclusion of (3)

$\therefore$ GC by modus ponens

Thus the glasses are on the coffee table.