EE320 Lecture Note
Chapter 9: Optimization under Equality Constraint

1 Effect of constraint

ex. $\max\limits_{x} y = -x^2 + 10x + 4$

$$\begin{aligned} \frac{dy}{dx} = -2x + 10 &= 0 \\ x^* &= 5 \\ y^* &= 29 \end{aligned}$$

$\frac{d^2y}{dx^2} = -2 < 0 \quad \text{max.}$

"unconstraint optimization"

ex. $\max\limits_{x} y = -x^2 + 10x + 4$ subject to x = 0

"constraint optimization"

x = 0⇒ y = 4

"equality constraint"

st x ≥ 2

"inequality constraint"

ex. 3-dimension figure

Note maximum number of constraint = number of independent variable

ex.

$y = -x^2 + 10x + 4$ $\Rightarrow$ 1 independent variable = 1 maximum constraint

$y = f(x_1, x_2)$ $\Rightarrow$ 2 independent variable = 2 maximum constraint

$y = f(x_1, x_2, \ldots, x_n)$ $\Rightarrow$ n independent variable = n maximum constraint

2 Constraint Optimization

ex. $U = x_1x_2 + 2x_1$ utility function

$4x_1 + 2x_2 = 60$ budget constraint

What will the optimal combination of goods x_1 and x_2 that give the highest utility subject to budget constraint be?

i) Substitution method

$$\begin{aligned}
4x_1 + 2x_2 &= 60 \Rightarrow x_2 = 30 - 2x_1 \text{ substitute into utility function } U = x_1x_2 + 2x_1 \\
\Rightarrow U &= x_1(30 - 2x_1) + 2x_1 \\
&= 32x_1 - 2x_1^2 \\
\underline{FONC} \quad \frac{dU}{dx_1} &= 32 - 4x_1^* = 0 \Rightarrow x_1^* = 8 \; and \; x_2^* = 14 \\
\underline{SOSC} \quad \frac{d^2U}{dx_1^2} &= -4 < 0 \text{ max.}
\end{aligned}$$

ii) Lagrange - multiplier method

(convert a constraint optimization problem into an unconstraint optimization problem)

$\mathcal{L} = x_1x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2)$ or

$\mathcal{L} = x_1x_2 + 2x_1 - \lambda(4x_1 + 2x_2 - 60) \quad \Rightarrow$ give the same answer

$\mathcal{L} = L(\lambda, x_1, x_2)$

$$\left.\begin{aligned}
\underline{FOC} : \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 : 60 - 4x_1 - 2x_2 = 0 - ① \\
\frac{\partial \mathcal{L}}{\partial x_1} &= 0 : x_2 + 2 - 4\lambda = 0 - ② \\
\frac{\partial \mathcal{L}}{\partial x_2} &= 0 : x_1 - 2\lambda = 0 - ③
\end{aligned}\right\}$$

3 unknowns, 3 equations solve for λ^*, x_1^*, x_2^*

$$② : x_2 + 2 = 4\lambda \quad --④ \quad [\text{Note} : x_2 + 2 = MU_1]$$

$$③ : x_1 = 2\lambda \quad --⑤ \quad [\text{Note} : x_1 = MU_2]$$

$$\frac{④}{⑤} : \frac{x_2+2}{x_1} = 2 \quad \Rightarrow \quad x_2 + 2 = 2x_1$$

$$2x_2 + 4 = 4x_1$$

Substitute in ①$(\frac{MU_1}{MU_2} = \frac{P_{x_1}}{P_{x_2}})$

$$\begin{aligned} \Rightarrow 60 - (2x_2 + 4) - 2x_2 &= 0 \\ 56 - 4x_2 &= 0 \\ x_2^* &= 14\ ,\ x_1^* = 8 \\ \lambda^* &= 4 \end{aligned}$$

3 Interpretation of lagrage multiplier

Consider the problem $\max_{x,y} f(x,y) st g(x,y) = c$

Solutions are $x^*(c), y^*(c), f^*(c)$

$f^*(c)$ is optimal value function AND $\frac{d}{dc}f^*(c) = \lambda(c)$

Lagrage multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant c.

In economic application, c denotes the available stock of some resources, f(x,y) denotes utility, profit, etc. $df^*(c) = \lambda(c)dc$ is the change in utility or profit that can ve obtained from dc units more of the resource.

Thus, λ is called a "shadow price" of the resource

ex. for cost minimization problem , $\lambda = \frac{dC}{dQ_0} = MC$

utility max. $\lambda = \frac{dC}{dY_0} = MU of income$

profit max. $\lambda = \frac{d\Pi}{dQ_0}$

output max. $\lambda = \frac{dQ}{dC_0}$

ex.

Obj. z = xy

st. x+y = 6

$\mathcal{L} = xy + \lambda(6 - x - y)$

$\Rightarrow x^* = 3, y^* = 3, \lambda^* = 3, z^* = 9$

ex.

Obj. $f(x_1, x_2) = x_1^2 + x_2^2$

st. $g(x_1, x_2) = x_1 + 4x_2 = 2$

$\mathcal{L} = x_1^2 + x_2^2 + \lambda(2 - x_1 - 4x_2)$

$\Rightarrow x_1^* = \frac{2}{17}, x_2^* = \frac{8}{17}$

4 Second-order conditions for constrained optimization

$$\begin{aligned}
\text{Recall the FONC for } L(x,y) &= f(x,y) + \lambda[c - g(x,y)] \\
L_x(x,y) &= f_x(x,y) - \lambda g_x(x,y) \\
L_y(x,y) &= f_y(x,y) - \lambda g_y(x,y) \\
\lambda &= \tfrac{f_x}{g_x} = \tfrac{f_y}{g_y}
\end{aligned}$$

Find SOSC

$$\begin{aligned}
L_{xx}(x,y) &= f_{xx} - \lambda g_{xx} = f_{xx} - \tfrac{f_y}{g_y} g_{xx} \\
L_{yy}(x,y) &= f_{yy} - \lambda g_{yy} = f_{yy} - \tfrac{f_y}{g_y} g_{yy} \\
L_{xy}(x,y) &= f_{xy} - \lambda g_{xy} = f_{xy} - \tfrac{f_y}{g_y} g_{xy}
\end{aligned}$$

Thus, d^2z can be written as

$$\begin{aligned}
d^2z &= L_{xx}dx^2 + 2L_{xy}dxdy + L_{yy}dy^2 \\
&= \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} L_{xx} & L_{xy} \\ L_{yx} & L_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}
\end{aligned}$$

SOSC conditions are:

Max of z : d^2z is negative definite (d^2z<0) , st. dg = 0

Min of z : d^2z is positive definite (d^2z>0) , st. dg = 0

where $dg = g_x dx + g_y dy = 0$

Define the determinant of the Bordered Hessian matrix as

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$$

or SOSC conditions are

Max: d^2z is negative definite st. $g_x dx + g_y dy = 0$ iff $|\bar{H}| > 0$

Min: d^2z is positive definite st. $g_x dx + g_y dy = 0$ iff $|\bar{H}| < 0$

ex.

opt. $U = x_1x_2 + 2x_1$

st. $4x_1 + 2x_2 = 60$

max or min ?

$\mathcal{L} = x_1x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2)$

FONC

$$\left.\begin{aligned} \mathcal{L}_1 &= x_2 + 2 - 4\lambda = 0 \\ \mathcal{L}_2 &= x_1 - 2\lambda = 0 \\ \mathcal{L}_\lambda &= 60 - 4x_1 - 2x_2 = 0 \end{aligned}\right\} x_1^* = 8\ ,\ x_2^* = 14 \ \Rightarrow\ U = 128$$

SOSC

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & +4 & +2 \\ +4 & +2 & 1 \\ +2 & 1 & 0 \end{vmatrix} = 16$$

$\therefore |\bar{H}| = 16 > 0$ maximum

$U^*(8, 14) = 128$ is a maximum utility subject to budget contraint.

ex. $f(x_1, x_2) = x_1^* + x_2^*$st. $g(x_1, x_2) = x_1 + 4x_2 = 2$

max,min?

SOSC $\quad |\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{vmatrix} = 34$

$|\bar{H}| = -34 < 0$ minimum

5 Economic Applications of Constrained Optimization

(Application: Utility maximization Problem)

Objective function: $\max\limits_{x_1,x_2} U = U(x_2, x_2)$

Subject to $y_0 = p_1x_1 + p_2x_2$

$\mathcal{L} = U(x_1, x_2) + \lambda(y_0 - p_1x_1 - p_2x_2)$

FONC

$$\begin{aligned} L_1 &= u_1(x_1, x_2) - \lambda p_1 &= 0 \\ L_2 &= u_2(x_1, x_2) - \lambda p_2 &= 0 \\ L_\lambda &= y_0 - p_1x_1 - p_2x_2 &= 0 \end{aligned}$$

$D(x_1^*, x_2^*)$ is such that $\frac{u_1(x_1,x_2)}{u_2(x_1,x_2)} = \frac{P_1}{P_2}$ and $y_0 = p_1x_1 + p_2x_2$

$$\text{from} \quad \left.\begin{aligned} \lambda &= \tfrac{u_1}{p_1} \\ \lambda &= \tfrac{u_2}{p_2} \end{aligned}\right\} \tfrac{u_1}{p_1} = \tfrac{u_2}{p_2} \quad \Rightarrow \quad \tfrac{u_1}{u_2} = \tfrac{p_1}{p_2}$$

SOSC $\quad |\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & u_{11} & u_{21} \\ p_2 & u_{21} & u_{22} \end{vmatrix} = -p_1^2u_{11}^2 + 2p_1p_2u_{12} - p_2^2u_{11}^2$

ex. $U = 50A + 200B - 0.5A^2 - 2.5B^2 \quad \Rightarrow \quad$ Find A,B that maximize the utility given

$$\left.\begin{array}{rcl} P_A & = & 10 \\ P_B & = & 5 \\ \bar{Y} & = & 490 \end{array}\right\} 490 = 10A + 5B$$

$\therefore \max U = 50A + 200B - 0.5A^2 - 2.5B^2$

st. 10A + 5B = 490

$$\begin{array}{rcl} A^* & = & 30 \\ B^* & = & 38 \\ U^*(A^*, B^*) & = & 5,040 \end{array}$$

$|\bar{H}| = 525 > 0$ max.

(Application: Expenditure Minimization Problem)

Objective function $\min\limits_{x_1,x_2} E = p_1x_1 + p_2x_2$

Subject to $\bar{U} = U(x_1, x_2)$

$\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(\bar{U} - U(x_1, x_2))$

FONC

$$\left.\begin{array}{rclcl} L_1 & = & p_1 - \lambda u_1(x_1, x_2) & = & 0 \\ L_2 & = & p_2 - \lambda u_2(x_1, x_2) & = & 0 \\ L_\lambda & = & \bar{u} - u(x_1, x_2) & = & 0 \end{array}\right\} \frac{u_1}{u_2} = \frac{p_1}{p_2} \; and \; \bar{u} = u(x_1, x_2)$$

SOSC $\quad |\bar{H}| < 0$ min.

ex. $U = 50A + 200B - 0.5A^2 - 2.5B^2$

and consumer would like to have a utility fixed at $\bar{u} = 5{,}040$

Find A and B that minimize the expenditure given that $P_A = 10$ and $P_B = 5$

Min $E = 10A + 5B$

st. $5,040 = 50A + 200B - 0.5A^2 - 2.5B^2$

$$\begin{aligned} \Rightarrow A^* &= 30 \\ B^* &= 38 \\ E^* &= 490 \end{aligned}$$

ex. $u = x_1 x_2 \qquad p_1, p_2, y_0$

$\max\limits_{x_1,x_2} u = x_1 x_2$

st. $p_1 x_1 + p_2 x_2 = y_0$

$\mathcal{L} = x_1 x_2 + \lambda(y_0 - p_1 x_1 - p_2 x_2)$

FOC:

$$\begin{aligned} \mathcal{L}_1 &= x_2 - \lambda p_1 = 0 & \Rightarrow\ & x_2 = \lambda p_1 \\ \mathcal{L}_2 &= x_1 - \lambda p_2 = 0 & \Rightarrow\ & x_1 = \lambda p_2 \\ \mathcal{L}_\lambda &= p_1 x_1 + p_2 x_2 = y_0 \end{aligned}$$

$\therefore \frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2} x_1$

$\Rightarrow p_1 x_1^* + p_2(\frac{p_1}{p_2} x_1^*) = y_0$

$2p_1 x_1^* = y_0$

$\Rightarrow x_1^* = \frac{I}{2p_1}$

$x_2^* = \frac{I}{2p_2}$

$$\min_{x_1,x_2} E = p_1x_1 + p_2x_2$$

st. $x_1 + x_2 = \bar{u}$

$$\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(\bar{u} - x_1x_2)$$

FOC:

$$\begin{array}{rcl} \mathcal{L}_1 & = & p_1 - \lambda x_2 = 0 \quad \Rightarrow \quad p_1 = \lambda x_2 \\ \mathcal{L}_2 & = & p_2 - \lambda x_1 = 0 \quad \Rightarrow \quad p_2 = \lambda x_1 \\ \mathcal{L}_\lambda & = & x_1x_2 = \bar{u} \end{array}$$

$\therefore \frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2}x_1$

$\Rightarrow \bar{u} = x_1^*(\frac{p_1}{p_2}x_1^*)$

$\Rightarrow x_1^* = \sqrt{(\frac{p_2}{p_1}\bar{u})}$

$x_2^* = \sqrt{(\frac{p_1}{p_2}\bar{u})}$

(Application: Profit Maximization Problem)

Objective function $\max_{q_1,q_2} \Pi = p_1q_1 + p_2q_2 - c(q_1, q_2)$

Subject to $\bar{Q}_0 = q_1 + q_2$

$$\mathcal{L} = p_1q_1 + p_2q_2 - c(q_1, q_2) + (\bar{Q} - q_1 - q_2)$$

<u>FONC</u>

$$\left.\begin{array}{l} L_1 = p_1 - c_1(q_1, q_2) - \lambda = 0 \\ L_2 = p_2 - c_2(q_1, q_2) - \lambda = 0 \end{array}\right\} p_1 - c_1 = p_2 - c_2 \text{ and } \bar{Q} = q_1 + q_2$$

$$L_\lambda = \bar{Q} - q_1 - q_2 = 0$$

<u>SOSC</u> $|\bar{H}|>0$ max.

ex. Suppose that $p_1 = 80$, $p_2 = 100$, $TC = 100 + 0.1q_1^2 + 0.2q_2^2$ and $q_1 + q_2 = 325$

Find q_1 and q_2 that maximize profit.

$\max\limits_{q_1,q_2} = 80q_1 + 100q_2 - 100 - 0.1q_1^2 - 0.2q_2^2$

st. $q_1 + q_2 = 325$

$\Rightarrow \ \mathcal{L} = 80q_1 + 100q_2 - 100 - 0.1q_1^2 - 0.2q_2^2 + \lambda(325 - q_1 - q_2)$

FONC

$$\left.\begin{array}{rcl} L_1 &=& 80 - 0.2q_1 - \lambda = 0 \\ L_2 &=& 100 - 0.4q_2 - \lambda = 0 \end{array}\right\} q_1 + q_2 = 100$$

$$L_\lambda = 325 - q_2 - q_2 = 0$$

$$q_1^* = \tfrac{550}{3} \ , \ q_2^* = \tfrac{125}{3}$$

SOSC $\quad |\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -4 \end{vmatrix} = 6 > 0$ max.

(Application: Output Maximization Problem)

Objective function $\max\limits_{K,L} Q = Q(K, L)$

Subject to $\bar{C} = wL + rK$

$\mathcal{L} = Q(K, L) + \lambda(\bar{C} - wL - rK)$

FONC

$$\left.\begin{array}{rcl} L_K &=& Q_K - \lambda r = 0 \\ L_L &=& Q_L - \lambda w = 0 \end{array}\right\} \tfrac{Q_K}{Q_L} = \tfrac{r}{w}$$

$$L_\lambda = C_0 - wL - rK = 0$$

$$Q^*(K^*, L^*) \ , \ K^* \ , \ L^*$$

SOSC $|\bar{H}| = \begin{vmatrix} 0 & r & w \\ r & Q_{KK} & Q_{LK} \\ w & Q_{KL} & Q_{LL} \end{vmatrix}$

<u>ex.</u> Suppose Q = KL , w = 6 , r = 10 , $\bar{C}$ = 60. Find K and L that maximize output.

$\max_{K,L} Q = KL$

st. 60 = 6L+10K

$\mathcal{L}$ = KL + λ (60-6L-10K) $\quad\Rightarrow\quad L^* = 5\ K^* = 3$

$|\bar{H}| = 120 > 0$ max.

(Application: Cost Minimization Problem)

Objective function $\min_{K,L} C = wL + rK$

Subject to $\bar{Q}$ = f(K,L)

$\mathcal{L} = wL + rK + \lambda(\bar{Q} - f(K,L))$

<u>FONC</u>

$$\left.\begin{array}{r} L_K = r - \lambda f_K = 0 \\ L_L = w - \lambda f_L = 0 \end{array}\right\} \frac{f_K}{f_L} = \frac{r}{w} \qquad \Rightarrow \qquad C^*(K^*, L^*)\ ,\ K^*\ ,\ L^*$$

$$L_\lambda = \bar{Q} - f(K,L) = 0$$

<u>SOSC</u> $|\bar{H}| = \begin{vmatrix} 0 & f_K & f_L \\ f_K & L_{KK} & L_{LK} \\ f_L & L_{KL} & L_{LL} \end{vmatrix} < 0$ min.

ex. Suppose Q = KL. A firm will produce 15 units of goods.

Find K and L that minimize total cost given that w = 6 , r = 10.

Min C(w,r) 6L +10K

st. 15 = KL

$\mathcal{L} = 6L + 10K + \lambda(15 - KL)$

$$\begin{aligned} \Rightarrow \quad K^* &= 3 \\ L^* &= 5 \\ |\bar{H}| &= -60 < 0 \end{aligned}$$

6 n-choice Variable and Multi-Constraint Cases

6.1 One Constraint

Objective function z = $f(x_1, x_2, ..., x_n)$

Subject to $g(x_1, x_2, ..., x_n) = c$ (one constraint)

$\mathcal{L}(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) + \lambda[c - g(x_1, x_2, ..., x_n)]$

FONC

$$\left.\begin{aligned} L_1 &= f_1 - \lambda g_1 = 0 \\ L_2 &= f_2 - \lambda g_2 = 0 \\ &\vdots \\ L_n &= f_n - \lambda g_n = 0 \\ L_\lambda &= c - g(x_1, x_2, ..., x_n) = 0 \end{aligned}\right\} \text{(n+1) conditions}$$

SOSC $\quad |\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}_{(n+1)\times(n+1)}$

d^2z is positive definite st. dg = 0 iff $|\bar{H}_2|, |\bar{H}_3|, ..., |\bar{H}_n| < 0$ always negative <u>Min</u>

negative definite st. dg = 0 iff $|\bar{H}_2| > 0$, <u>Max</u>

$$|\bar{H}_3| < 0$$

$$\vdots$$

$$(-1)^n|\bar{H}_n| > 0 \quad \text{alternate signs}$$

<u>ex.</u> min C = wL +iK + rT   find K , L and T that minimize cost

st. $\bar{Q}$ = f(K,L,T)

$\mathcal{L} = wL + iK + rT + \lambda(\bar{Q} - f(K, L, T))$

$$\vdots$$

6.2 k Equality Constraints

Objective function $z = f(x_1, x_2, ..., x_n)$

Subject to constraints

$$\begin{array}{rcl} g^1(x_1, x_2, ..., x_n) & = & c_1 \\ g^2(x_1, x_2, ..., x_n) & = & c_2 \\ \vdots & & \vdots \\ g^k(x_1, x_2, ..., x_n) & = & c_k \end{array}$$

$$\mathcal{L} = f(x_1, x_2, ..., x_n) + \lambda_1[c_1 - g^1(\cdot)] + \lambda_2[c_2 - g^2(\cdot)] + \cdots + +\lambda_k[c_k - g^k(\cdot)]$$

$$\mathcal{L} = f(x_1, x_2, ..., x_n) + \sum_{j=1}^{k} \lambda_j[c_j - g^j(x_1, x_2, ..., x_n)]$$

<u>FONC</u>

$$\left.\begin{array}{r} \frac{\partial L}{\partial x_1} = f_1 - \lambda_1 \frac{\partial g^1}{\partial x_1} - \lambda_2 \frac{\partial g^2}{\partial x_1} - \cdots - \lambda_k \frac{\partial g^k}{\partial x_1} = 0 \\ \vdots \qquad\qquad \vdots \\ \frac{\partial L}{\partial x_n} = f_n - \lambda_1 \frac{\partial g^1}{\partial x_n} - \lambda_2 \frac{\partial g^2}{\partial x_n} - \cdots - \lambda_k \frac{\partial g^k}{\partial x_n} = 0 \end{array}\right\} \text{n equations}$$

$$\left.\begin{array}{r} \frac{\partial L}{\partial \lambda_1} = c_1 - g^1(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ \frac{\partial L}{\partial \lambda_k} = c_k - g^k(x_1, x_2, ..., x_n) = 0 \end{array}\right\} \text{k equations}$$

<u>SOSC</u>

$$\bar{H} = \begin{vmatrix} 0 & 0 & \cdots & 0 & | & g_1^1 & g_2^1 & \cdots & g_n^1 \\ 0 & 0 & \cdots & 0 & | & g_1^2 & g_2^2 & \cdots & g_n^2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & | & g_1^k & g_2^k & \cdots & g_n^k \\ - & - & - & - & - & - & - & - & - \\ g_1^1 & g_2^1 & \cdots & g_n^1 & | & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2^1 & g_2^2 & \cdots & g_2^k & | & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & | & \vdots & \vdots & \ddots & \vdots \\ g_n^1 & g_n^2 & \cdots & g_n^k & | & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}_{(n+k)\times(n+k)}$$

Consider (n-k) bordered leading principal minors:

$|\bar{H}_{k+1}|, |\bar{H}_{k+2}|, \ldots, |\bar{H}_n|$

Max of z, sign $|\bar{H}_{k+1}| = (-1)^{k+1}$ alternate sign for $|\bar{H}_{k+2}|$ and so on.

Min of z, sign $|\bar{H}_k| = (-1)^{k+1}$ take the same sign

<u>ex.</u>

$z = f(x_1, x_2, x_3)$

st. $g(x_1, x_2, x_3) = c$ and $h(x_1, x_2, x_3) = d$

$\mathcal{L} = f(x_1, x_2, x_3) + \lambda_1[c - g] + \lambda_2[d - h]$

<u>FONC</u>:

<u>SOSC</u>: n = 3, k= 2 $\Rightarrow$ n-k = 1 only one bordered Hessian matrix

$|\bar{H}_n| = |\bar{H}_3|$ or $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_n|$

$$\therefore |\bar{H}_3| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_3 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}_{5\times5}$$

max if $|\bar{H}_{k+1}| = |\bar{H}_3| < 0$ because sign of $|\bar{H}_3| = (-1)^{2+1} = -1 < 0$ should be negative

max if $|\bar{H}_{k+1}| = |\bar{H}_3| > 0$ because sign of $|\bar{H}_3| = (-1)^2 = 1 > 0$ should be positive

<u>ex.</u> $z = f(x_1, x_2, x_3, x_4)$

st. $g(x_1, x_2, x_3, x_4) = c$

$h(x_1, x_2, x_3, x_4) = d$

<u>FONC</u>:

<u>SOSC</u> n = 4, k =2 $\Rightarrow$ n - k = 2 bordered Hessian matrix.

Consider $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_3| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_3 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}$

$$|\bar{H}_{k+2}| = |\bar{H}_{2+2}| = |\bar{H}_4| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & h_1 & h_2 & h_3 & h_4 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} & L_{14} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} & L_{24} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} & L_{34} \\ g_4 & h_4 & L_{41} & L_{42} & L_{43} & L_{44} \end{vmatrix}_{6\times 6}$$

max if sign of $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_3| = (-1)^{2+1} = -1$ negative
or $|\bar{H}_3| < 0$ and $|\bar{H}_{k+2}| = |\bar{H}_4| > 0$

min if sign of $|\bar{H}_3| = (-1)^2 = 1$ or $|\bar{H}_3|$>0 and $|\bar{H}_{k+2}| = |\bar{H}_4| > 0$

<u>ex.</u> Given objective function $y = f(x_1, \ldots, x_{35})$ and 22 constraints

1. What is the dimension of $\bar{H}$? =(n+k)×(n+k) = (22+35)×(22+35) = (57× 57)

2. What is the dimension of zero matrix inside $\bar{H}$? = k × k = 22 × 22

3. Write SOCs for max. = $|\bar{H}_{23}| < 0$, $|\bar{H}_{24}| > 0$, $|\bar{H}_{25}| < 0$, keep alternating signs

7 Other Economic Application

7.1 Homogeneous function

$F(\lambda x, \lambda y) = \lambda^r F(x, y)$

where r is a degree of homogeneous function.

indicates 3 types of return to scales(IRTS, CRTS, DRTS)

<u>ex.</u> Q = KL

$K_0,\ L_0 \quad \Rightarrow \quad Q_0 = K_0 L_0$

$\lambda K_0, \lambda L_0 \ \Rightarrow \ Q_1 = \lambda K_0 \lambda L_0 = \lambda^2 K_0 L_0 = \lambda^2 Q_0$

If r = 2, then if we increase both L and K by 2 times,

then the level of output would be rising by $\lambda^2 = 2^2 = 4$ times = IRTS

<u>ex.</u> Q =K+L

$Q(\lambda K, \lambda L) = \lambda K_0 + \lambda L_0 = \lambda(K + L) = \lambda Q_0(K, L) \ \Rightarrow \quad$ CRTS

<u>ex.</u> $Q = K^\alpha L^\beta$

What should the value of α abd β be such that Q has a property of CRTS?

$Q(K, L) = K^\alpha L^\beta$

$$\begin{aligned} Q(\lambda K, \lambda L) &= (\lambda K)^\alpha (\lambda L)^\beta \\ &= \lambda^{\alpha+\beta} K^\alpha L^\beta \\ &= \lambda^{\alpha+\beta} Q(K, L) \end{aligned}$$

$\alpha + \beta < 1$ DRTS
$\alpha + \beta = 1$ CRTS
$\alpha + \beta > 1$ IRTS

Recall

AC = $\frac{TC}{Q}$

at IRTS, $\uparrow$ TC by 2, bit $\uparrow$ Q $> 2 \Rightarrow$ AC $\downarrow$

DRTS, $\uparrow$ TC by 2, bit $\uparrow$ Q $< 2 \Rightarrow$ AC $\uparrow$

ex $TC = 2\sqrt{Q}$

AC = $\frac{TC}{Q} = \frac{2\sqrt{Q}}{Q} = \frac{2}{\sqrt{Q}}$ IRTS $\therefore Q \uparrow \Rightarrow AC \downarrow$

$$\therefore F(\lambda x, \lambda y) = \lambda^r F(x,y) \quad \text{if} \quad \begin{array}{lclcl} r = 1 & , & CRTS & \Rightarrow & \text{AC constant} \\ r > 1 & , & \text{IRTS} & \Rightarrow & \text{AC} \downarrow \\ r < 1 & , & \text{DRTS} & \Rightarrow & \text{AC} \uparrow \end{array}$$

7.2 Types of Goods

$$\underline{\text{ex.}} \quad \left.\begin{array}{rcl} U & = & x + y \\ P_x & = & 2 \\ P_y & = & 1 \\ I & = & 100 \end{array}\right\} \text{(1st approach) } \mathcal{L} = x + y + \lambda(100 - 2x - y)$$

FONC

$$\left.\begin{array}{rclcl} \mathcal{L}_\lambda & = & 100 - 2x - y & = & 0 \\ \mathcal{L}_x & = & 1 - 2\lambda & = & 0 \\ \mathcal{L}_y & = & 1 - \lambda & = & 0 \end{array}\right\} \text{Lagragian cannot solve the corner solution.}$$

$$\text{(2nd approach)} \quad \begin{array}{rcl} MU_x & = & 1 \\ MU_y & = & 1 \\ MRS & = & 1 \end{array}$$

$\therefore$ should buy more Y $\because$ it is cheaper and give same MU

$\Rightarrow$ Demand for x = 0 and Demand for y = 100

This case $\frac{P_x}{P_y} = \frac{2}{1} = 1 > |MRS|$ $\Rightarrow$ buy only y since y is cheaper

How about $|\frac{P_x}{P_y}| < |MRS|$ $\Rightarrow$ buy only x since x is cheaper

$|\frac{P_x}{P_y}| = |MRS|$ $\Rightarrow$ choose any bundles on BC

ex. U = min [x,y]

x	y	u
1	1	1
1	2	1
2	1	1
2	3	2

"perfect complement"

ex. U = min [2x,y]

kink point is 2x = y

ex.
$$\left.\begin{array}{rcl} U & = & \min[x,y] \\ P_x & = & 2 \\ P_y & = & 1 \\ I & = & 100 \end{array}\right\}$$

(1st approach) $\mathcal{L} = \min[x,y] + \lambda(100 - 2x - y)$, does not work.

Solve for x^*, y^*

(2nd approach)

kink point x = y

and BC: 100-2x-y = 0

Solve for x and y $\Rightarrow$ 100 -x -2x = 0

$$x^* = \frac{100}{3}\ ,\ y^* = \frac{100}{3}$$

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