Answer

1. a. Derive factor inputs demand

Total cost function $TC = K \cdot r + L \cdot w$

min. $TC = Kr + Lw$

s.t $Q_0 = \alpha\sqrt{K} + \beta L$

Lagrange Equation

$$\mathcal{L} = Kr + Lw + \lambda[Q_0 - \alpha\sqrt{K} - \beta L]$$

F.O.C : $\frac{\partial \mathcal{L}}{\partial K} = 0 \iff r - \alpha\lambda\frac{1}{2}K^{-1/2} = 0 \Rightarrow r = \alpha\lambda\frac{1}{2}K^{-1/2}$ ①

$\frac{\partial \mathcal{L}}{\partial L} = 0 \iff w - \lambda\beta = 0 \Rightarrow w = \beta\lambda$ ②

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff Q_0 - \alpha\sqrt{K} - \beta L = 0$ ③

$$\frac{①}{②} = \frac{r}{w} = \frac{\alpha\lambda\frac{1}{2}K^{-1/2}}{\beta\lambda}$$

$$\Rightarrow \frac{1}{\sqrt{K}} = \frac{r\beta}{w\alpha\frac{1}{2}} = \frac{2r\beta}{w\alpha}$$

$$\sqrt{K} = \frac{w\alpha}{2r\beta}$$

$$\Rightarrow K^* = \left(\frac{w\alpha}{2r\beta}\right)^2$$

plug in K^* into ③ : $Q_0 - \alpha\sqrt{\left(\frac{w\alpha}{2r\beta}\right)^2} - \beta L = 0$

$$\beta L = Q_0 - \frac{w\alpha^2}{2r\beta}$$

$$\Rightarrow L^* = \frac{w\alpha^2}{2r\beta^2} + \frac{Q_0}{\beta}$$

b. Second order derivative test.

S.O.C

$$\bar{H} = \begin{bmatrix} 0 & -\frac{1}{2}\alpha K^{-1/2} & -\beta \\ -\frac{1}{2}\alpha K^{-1/2} & \frac{1}{4}\alpha K^{-3/2} & 0 \\ -\beta & 0 & 0 \end{bmatrix}$$

The last 2-1 submatrices = last 1 submatrix = $\bar{H}_3$

$$|H_3| = \begin{bmatrix} 0 & -\frac{1}{2}\alpha K^{-1/2} & -\beta \\ -\frac{1}{2}\alpha K^{-1/2} & \frac{1}{4}\alpha K^{-3/2} & 0 \\ -\beta & 0 & 0 \end{bmatrix} \begin{matrix} 0 & -\frac{1}{2}\alpha K^{-1/2} \\ -\frac{1}{2}\alpha K^{-1/2} & \frac{1}{4}\alpha K^{-3/2} \\ -\beta & 0 \end{matrix}$$

$$= 0+0+0-\left(\beta^2\frac{1}{4}\alpha K^{-3/2}+0+0\right) = -\beta^2\frac{1}{4}\alpha K^{-3/4} < 0$$

Therefore, solution from F.O.C minimized objective function (Total cost).

$\therefore$ Factor inputs demand : $K^* = \left(\frac{w\alpha}{2r\beta}\right)^2$ and $L^* = \frac{w\alpha^2}{2r\beta^2} + \frac{Q_0}{\beta}$

c. State the condition under which demand for capital and labor are both strictly positive.

$K^* = \left(\frac{w\alpha}{2r\beta}\right)^2 > 0$; K^* will always be positive for are variables w, α, r, β.

$$L^* = \frac{w\alpha^2}{2r\beta^2} + \frac{Q_0}{\beta}$$

L^* will be positive if $w \geq 0$, $r > 0$, $\beta \neq 0$

$\therefore$ Therefore, $w \geq 0$, $r > 0$, $\beta \neq 0$, α can be any numbers.

d. Derive the long-run optimal cost function

$$TC(K^*, L^*) = \left(\frac{w\alpha}{2r\beta}\right)^2 r + \left(\frac{w\alpha^2}{2r\beta^2} + \frac{Q_0}{\beta}\right) w$$

MC : $\frac{\partial TC}{\partial Q_0} = \frac{w}{\beta}$

Lagrange multiplier λ^* : $\frac{\partial TC}{\partial \text{constraint}} = \frac{\partial TC}{\partial Q_0}$

Derive λ^* from question a

from ② : $w = \beta\lambda \Rightarrow \lambda^* = \frac{w}{\beta}$

$\therefore$ Therefore $MC = \lambda^* = \frac{w}{\beta}$

2\. a. Calculate total differential of utility function

$$\frac{\partial U}{\partial x} = \frac{1}{2}[x^2+y^2]^{-1/2} 2x = x[x^2+y^2]^{-1/2}$$

$$\frac{\partial U}{\partial y} = \frac{1}{2}[x^2+y^2]^{-1/2} 2y = y[x^2+y^2]^{-1/2}$$

Total differential $\partial U = x[x^2+y^2]^{-1/2} dx + y[x^2+y^2]^{-1/2} dy$

b. Set up constrained optimization problem and derive Marshallian demand function

Max. $U(x,y) = [x^2+y^2]^{\frac{1}{2}}$

s.t $P_x \cdot x + P_y \cdot y = M$

Lagrange Equation:

$$\mathcal{L} = [x^2+y^2]^{\frac{1}{2}} + \lambda[M - P_x \cdot x - P_y \cdot y]$$

F.O.C : $\frac{\partial x}{\partial \mathcal{L}} = 0 \Leftrightarrow x[x^2+y^2]^{-\frac{1}{2}} - P_x\lambda = 0 \Rightarrow x[x^2+y^2]^{-\frac{1}{2}} = P_x\lambda$ ①

$\frac{\partial y}{\partial \mathcal{L}} = 0 \Leftrightarrow y[x^2+y^2]^{-\frac{1}{2}} - \lambda P_y = 0 \Rightarrow y[x^2+y^2]^{-\frac{1}{2}} = P_y\lambda$ ②

$\frac{\partial \lambda}{\partial \mathcal{L}} = 0 \Leftrightarrow M - P_x \cdot x - P_y \cdot y = 0$ ③

$\frac{①}{②}$: $\frac{x[x^2+y^2]^{-\frac{1}{2}}}{y[x^2+y^2]^{-1/2}} = \frac{P_x\lambda}{P_y\lambda}$

$$\frac{x}{y} = \frac{P_x}{P_y} \Rightarrow x = \frac{P_x}{P_y} y \quad ④$$

plug ④ into ③ : $M - P_x\left[\frac{P_x}{P_y} \cdot y\right] - P_y \cdot y = 0$

$$M - \frac{P_x^2}{P_y} \cdot y - P_y \cdot y = 0$$

$$y\left(\frac{P_x^2}{P_y} + P_y\right) = M \Rightarrow y^* = \frac{M}{\frac{P_x^2}{P_y} + P_y}$$

From ④ : $y = \frac{P_y}{P_x} \cdot x$ ⑤

plug ⑤ into ③ : $M - P_x \cdot x - P_y \cdot \frac{P_y}{P_x} \cdot x = 0$

$$P_x \cdot x + \frac{P_y^2}{P_x} x = M$$

$$\Rightarrow x^* = \frac{M}{P_x + \frac{P_y^2}{P_x}}$$

Marshallian demand function: $x^* = \frac{M}{P_x + \frac{P_y^2}{P_x}}$, $y^* = \frac{M}{\frac{P_x^2}{P_y} + P_y}$

c.

$$x^* = \frac{M}{\frac{P_x^2 + P_y^2}{P_x}} = \frac{MP_x}{P_x^2 + P_y^2}$$

$$\frac{\partial x^*}{\partial P_x} = \frac{M(P_x^2 + P_y^2) - 2P_x(MP_x)}{(P_x^2 + P_y^2)^2} = \frac{M[P_x^2 + P_y^2 - 2P_x^2]}{(P_x^2 + P_y^2)^2} = \frac{-M(P_x^2 - P_y^2)}{(P_x^2 + P_y)^2} < 0$$

$$y^* = \frac{M}{\frac{P_x^2 + P_y^2}{P_y}} = \frac{MP_y}{P_x^2 + P_y^2}$$

$$\frac{\partial y^*}{\partial P_y} = \frac{M(P_x^2 + P_y^2) - MP_y 2P_y}{(P_x^2 + P_y^2)^2} = \frac{M(P_x^2 + P_y^2 - 2P_y^2)}{(P_x^2 + P_y^2)^2} = \frac{-M(P_y^2 - P_x^2)}{(P_x^2 + P_y^2)^2} < 0$$

d.

$$y^* = \frac{MP_y}{P_x^2 + P_y^2}$$

$$\frac{\partial y^*}{\partial P_x} = \frac{-2P_x \cdot MP_y}{P_x^2 + P_y^2} < 0$$

When price of x increase by 1 unit, demand for good Y decreases by $\frac{2MP_xP_y}{(P_x^2 + P_y^2)^2}$ unit

Good x and good Y are complementary products.

e. Find λ^*

From question b: plug in x^* & y^* to ①: $\frac{MP_x}{P_x^2 + P_y^2}\left[\left(\frac{MP_x}{P_x^2 + P_y^2}\right)^2 + \left(\frac{MP_y}{P_x^2 + P_y^2}\right)^2\right]^{-1/2} = P_x \lambda$

Since $M = 300$; $P_x = 1$, $P_y = 1$

$$\Rightarrow \frac{300}{1+1}\left[\left(\frac{300}{1+1}\right)^2 + \left(\frac{300}{1+1}\right)^2\right]^{-\frac{1}{2}} = \lambda$$

$\Rightarrow \lambda^* = 0.71$

f. When constraint $M = 300$, $\lambda^* = 0.71$

When $M = 310 \Rightarrow \Delta M = 10$

$\Rightarrow \Delta U = \lambda^* \Delta M$

$= 0.71 \cdot 10 = 7.1$

Original optimized level of maximium utility

$$U(x^*, y^*) = \left[\left(\frac{MP_x}{P_x^2 + P_y^2}\right)^2 + \left(\frac{MP_y}{P_x^2 + P_y^2}\right)^2\right]^{1/2} = \left[\left(\frac{300}{1+1}\right)^2 + \left(\frac{300}{1+1}\right)^2\right]^{\frac{1}{2}} = 212.13$$

The new optimized level of utility $= 212.13 + \Delta U = 212.13 + 7.1 = 219.23$

The new optimized level of utility is higher than the previous one by $\Delta U = 7.1$.

3. Calculate the total and the average cost when $q = 9$ units

Given $MC = 16 + 6q^2$

$$\frac{dTC}{dq} = 16 + 6q^2$$

$$\int dTC = \int (16 + 6q^2)\,dq$$

$$TC = 16q + 2q^3 + C$$

When $q = 0 \Rightarrow TC = 240$

$$\Rightarrow 240 = 16 \cdot (0) + 2 \cdot 0 + C$$

$$\Rightarrow C = 240$$

$$TC(q=9) = 16(9) + 2(9)^3 + 240$$

$$= 1842 \text{ units}$$

$$AC = \frac{TC}{q} = 16 + 2q^2 + 240$$

$$AC(q=9) = 16 + 2(9)^2 + 240 = 418 \text{ units}$$

b. Determine the profit-maximizing level of output

$$\pi = TR - TC$$

$$= (160 - 10q^2)q - (16q + 2q^3 + 240)$$

$$= 160q - 10q^3 - 16q - 2q^3 - 240$$

$$= 144q - 12q^3$$

F.O.C : $\frac{\partial \pi}{\partial q} = 0 \Leftrightarrow 144 - 36q^2 = 0$

$$q^2 = \frac{144}{36} = 4$$

$$q^* = \sqrt{4} = 2$$

S.O.C : $\frac{\partial^2 \pi}{\partial q^2} = -72q < 0$ when $q \in [0, +\infty] \Rightarrow$ profit function concave when q is positive

Therefore, $q^* = 2$ is true maximized solution.

c. Calculate the social welfare under monopoly environment

- Monopolist will allocate P & q at its max. level of profit, so $q^* = 2$ units

$P^* = 160 - 10(2)^2 = 120$ units

Find CS

$$CS^m = \int_0^2 (160 - 10q^2)\,dq - (2 \times 120)$$
$$= \left[160q - 10\frac{q^3}{3}\right]_0^2 - 240$$
$$= 320 - \frac{80}{3} - 240 = 80 - \frac{80}{3} = \frac{160}{3} = 53.33$$
$$PS^m = (2 \times 120) - \int_0^2 (16 + 6q^2)\,dq$$
$$= 240 - \left[16q + 2q^3\right]_0^2 = 240 - 48 = 192$$
$$TS^m = 53.33 + 192 = 245.33$$

d. Calculate social welfare loss under the monopoly environment

In perfect competition market, $P = MC$

Equilibrium condition for PC : $P_s = P_d$

$$16 + 6q^2 = 160 - 10q^2$$
$$16q^2 = 144$$
$$q^2 = \frac{144}{16} = 9$$
$$q^* = 3 \text{ units}$$

$\Rightarrow P^* = 160 - 10 \cdot 3^2 = 70$ units

$$CS^{PC} = \int_0^3 (160 - 10q^2)\,dq - (3 \times 70) = \left[160q - 10\frac{q^3}{3}\right]_0^3 - 210$$
$$= 480 - 90 - 210 = 180$$
$$PS^{PC} = (3 \times 70) - \int_0^3 (16 + 6q^2)\,dq = 210 - \left[16q + 2q^3\right]_0^3 = 210 - 102 = 108$$
$$TS^{PC} = 180 + 108 = 288$$

$DW = 288 - 245.33 = 42.67$

∴ Therefore, social welfare loss under monopoly environment = 42.67 units.

4. Find Equilibrium price and quantity

$$P_s = P_d \iff \frac{6000}{Q+50} = Q+10$$

$$(Q+10)(Q+50) = 6000$$

$$Q^2 + 50Q + 10Q + 500 = 6000$$

$$Q^2 + 60Q - 5500 = 0$$

$$(Q+110)(Q-50) = 0 \Rightarrow Q_1 = -110;\ Q_2 = 50$$

$\Rightarrow Q^* = 50$

$\Rightarrow P^* = 50 + 10 = 60$

Find CS

$$CS = \int_0^{50} \left(\frac{6000}{Q+50}\right) dQ - (50 \times 60) = \Big[6000 \ln|Q+50|\Big]_0^{50} - 3000$$

$$= 6000 \ln 100 - 6000 \ln 50 - 3000 = 1158.88$$

$$PS = (50 \times 60) - \int_0^{50} (Q+10)\, dQ = 3000 - \left[\frac{Q^2}{2} + 10Q\right]_0^{50} = 3000 - 1750 = 1250$$

5. Determine the level of production that maximizes profit

profit-max. when $MR = MC$

$$25 - 5x - 2x^2 = 10 - 3x - x^2$$

$$x^2 + 2x - 15 = 0$$

$$(x+5)(x-3) = 0 \Rightarrow x_1 = -5,\ x_2 = 3$$

$\Rightarrow x^* = 3$ is level of output that maximize profit.

Determine the level of maximized profit

$\pi = TR - TC$

$\pi = \int MR - \int MC$

$$= \int (MR - MC) = \int (25 - 5x - 2x^2 - 10 + 3x + x^2)\, dx$$

$$= 25x - 5\frac{x^2}{2} - \frac{2x^3}{3} - \left(10x - \frac{3x^2}{2} - x^3 + C\right)$$

$(\pi = 0) = 7 \iff C = 7$

$$(\pi = 3) = 25 \cdot 3 - 5\frac{3^2}{2} - \frac{2 \cdot 3^3}{3} - \left(10 \cdot 3 - \frac{3 \cdot 3^2}{2} - 3^3 + 7\right) = 22$$