Topic 4 : Capital Asset Pricing Model and Abitrage Pricing Theory

EE431/438
Summary

Copeland, Thomas E. and J. Fred Weston, Financial Theory and Corporate Policy (4th ed), Addison-Wesley, 2005: Ch6 (pp 147 -157)
Federic Mishkin, The Economics of Money, Banking and Financial Markets (Appendix to Chapter 5, available in the internet):

October 2012

- CML : $E(R_p) = R_f + \left(\frac{E(R_m) - R_f}{\sigma_m} \right) \sigma_p$
- CML shows the relationship between expected return and “total” risk for efficiently diversified portfolios. Investors hold portfolios along the CML. Their market required rate of returns are determined by the relationship between the market required rate of return and standard deviation, as shown by CML equation.
- Portfolios which do not lie on the CML are inefficient. Though we know the variance and the rate of return on those portfolios, we cannot be sure at what rate of return that the market will require from them in order to hold them in equilibrium. The variance may not be the correct measure of riskiness for an individual asset.

- Hence, we cannot use our knowledge of the mean and the variance of an individual asset to determine its market required rate of return.
- We derive the CAPM by equating the slope of CML to the slope of efficient frontier at point M.
- SML : $E(R_i) = R_f + ((E(R_m) - R_f)\,\beta_i$; $\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{COV(R_m, R_i)}{VAR(R_m)}$
- SML : "equilibrium relationship" between expected return and β - "*Systematic Risk*".
- Only the portion of total variance that is correlated with the economy(covariance risk) is relevant. Any portion of total risk that is not correlated with the economy is irrelevant and can be avoided at zero cost through diversification. We need the information on how the rate of return on an asset is correlated with the economy(covariance risk, systematic risk, β) to determine the market required rate of return on an asset.

- R_i is a random variable. $E(R_i)$ is a constant.
- R_m is a random variable. $E(R_m)$ is a constant.
- R_f is a contant.
- σ_i , σ_m, σ_{im} are a constant.
- According to the CAPM, $R_i = a_i + \beta_i(R_m - R_f) + \epsilon_i$.
- $E(R_i) = a_i + \beta_i(E(R_m) - R_f)$; $a_i = R_f - \beta_i R_f$ and $E(\epsilon_i) = 0$
- a_i is a constant.
- $R_i = a_i + \beta_i(R_m - R_f) + \epsilon_i$
- $\sigma_i^2 = \beta_i^2 \sigma_{im}^2 + \sigma_{\epsilon_i}^2$
- Total risk = systematic risk + unsystematic risk.
- All assets will fall on SML in equilibrium.
- $\beta_p = a\beta_X + b\beta_Y$

The Abitrage Pricing Theory

- Arbitrage : an abitrage opportunity arises when an investor can construct a zero investment portfolio that will yield a sure profit (risk-free)
- The CAPM : $R_i = R_f + \beta_i(R_m - R_f) + \epsilon_i$: only one source of systematic risk, systematic risk can be eliminated through diversification
- The APT: several sources of risk that cannot be eliminated through diversification
- The sources of risk : inflation, aggregate output, .. etc.
- $R_i = \beta^i_0 + \beta^i_1(\text{Factor 1}) + \beta^i_2(\text{Factor 2}) + ... + \beta^i_k(\text{Factor k}) + \epsilon^i$
- If there is only one factor and that factor is R_m, the APT is the same as the CAPM.
- the APT is more general than the CAPM

- CAPM: $E(R_i) = a_i + \beta_i(E(R_m) - R_f))$;
  $E(R_i) = \text{constant}_1 + \beta_i(\text{constant}_2) = a_i + \beta_i E(R_m)$
- APT :
  $E(R_i) = \beta_0^i + \beta_1^i(\text{Factor 1}) + \beta_2^i(\text{Factor 2}) + ... + \beta_k^i(\text{Factor k})$
- APT : $R_i = \beta_0^i + \beta_1^i F_1 + \beta_2^i F_2 + ... + \beta_k^i F_K$
- Assets with the same values of β_j for all factors j must have the same rate of returns.
- Example : $R_a = 0.08 + 0.6F_1$, $R_b = 0.02 - 0.2F_1$.
  - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has $\beta_1 = 1$.
  - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has $\beta_1 = 0$.
  - Determine the rate of returns of portiolio i which has any value of β_1^i.
  - $E(R_i) = R_f + (E(R_m) - R_f)\beta_1^i$; $R_m =$ the rate of return of the portfolio which has $\beta_1 = 1$, R_f = the rate of return of the portfolio which has $\beta_1 = 0$.

- At equilibrium, there must be no abitrage opportunity.
- Examples: 2 Factors
  $R_a = 0.10F_1 - 0.5F_2$
  $R_b = 0.08 + 2F_1 + F_2$
  $R_c = 0.05 + 0.5F_1 + 0.5F_2$
  $R_p = w_aR_a + w_bR_b + w_cR_c$
- Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has no risk
  $0.10w_a + 2w_b + 0.5w_c = 0$
  $-0.5w_a + 1w_b + 0.5w_c = 0$
  $w_a = \frac{5}{13},\ w_b = -\frac{3}{13},\ w_c = \frac{11}{13}$

- Determine the rate of return on portfolio which has $\beta_1 = 1$ and $\beta_2 = 0$.
- Determine the rate of return on portfolio which has $\beta_1 = 0$ and $\beta_2 = 1$
- $E(R_i) = R_f + (\lambda_1 - R_f)\beta_1^i + (\lambda_2 - R_f)\beta_2^i$; λ_j = the expected rate of return on portfolio which has $\beta_j = 1$ and other betas equal to zero.
- APT :
  $E(R_i) = \beta_0^i + \beta_1^i(\text{Factor 1}) + \beta_2^i(\text{Factor 2}) + ... + \beta_k^i(\text{Factor k})$
- $E(R_i) = R_f + (\lambda_1 - R_f)\beta_1^i + (\lambda_2 - R_f)\beta_2^i + ... + (\lambda_k - R_f)\beta_k^i$;
- λ_j = the expected rate of return on portfolio with unit sensitivity to the *j* the factor and zero sensitivity to all other factors.