6 Integration

6.1 Definition of Integral

For a function, $F(x)$ that its derivative is $f(x)$

Or mathematically $\qquad F'(x) = f(x)$

This function, $F(x)$, then can be called an '**anti-derivative**' of $f(x)$. For example,

- If $f(x) = nx^{n-1}$, then $F(x) = x^n$; as long as $n \neq 0$.
- If $f(x) = Ce^{Cx}$, then $F(x) = e^{Cx}$.
- If $f(x) = \frac{1}{x}$, then $F(x) = \ln x$.

It should be noted that a function does not have a **unique** anti-derivative.

For example, if $\qquad f(x) = 3x^2$

then $\qquad F(x) = x^3$ is one anti-derivative

or $F(x) = x^3 + 1$, $F(x) = x^3 + 8$, …… etc ……. (Infinite number of anti-derivatives)

We may spot that any $F(x) = x^3 + C$ where C is a constant, can be anti-derivative of $f(x)$.

(You can try differentiate $F(x)$ and you will see that this is true because $\frac{d}{dx}(C) = 0$.)

Hence, for this fuction $f(x) = 3x^2$,

$F(x) = x^3 + C$ where C is a constant and is called **a constant of integration**.

This process of anti-differentiation is called **INTEGRATION**.

In other words, to integrate a function, $f(x)$ is to find $F(x)$ such that $F'(x) = f(x)$.

A symbol for integrating a function $f(x)$ is written as $\int f(x)dx$

$$\int f(x)dx = F(x) + C \quad \text{if and only if} \quad F'(x) = f(x)$$

where $\int$ is called the *integral sign.*

$f(x)$ is called the *integrand*.

C is called the *constant of integration*.

dx is part of the integral notation and indicates the variable involved.

REMEMBER that the integration is not unique; **DON'T FORGET THE CONSTANT OF INTEGRATION!**

6.2 Indefinite Integral

If we have to integrate a function, $f(x)$ and values of x are not given, we said we have to integrate without a limit. Hence, the integral will be a function of x.

From some of the differentiation rules, the following integration formulas can be derived:

1	$\int kdx = kx + C$, k is a constant
2	$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
3	$\int e^x dx = e^x + C$
4	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
5	$\int kf(x)dx = k\int f(x)dx$, k is a constant
6	$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

NOTE: we can always check the integration result by differentiating it!

Ex.1 Find the following integrals.

(i) $\int 5dx$ [ANS: $5x + C$]

(ii) $\int x^4 dx$ [ANS: $\frac{x^5}{5} + C$]

(iii) $\int \frac{1}{\sqrt{x}} dx$ [ANS: $2\sqrt{x} + C$]

(iv) $\int \left(2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1\right) dx$ [ANS: $\frac{10}{9}x^{\frac{9}{5}} - \frac{7}{4}x^4 + 10e^x - x + C$]

(v) $\int \left(\frac{(x-1)(2x+7)}{3} \right) dx$ [ANS: $\frac{1}{3}\left[\frac{2}{3}x^3 + \frac{5}{2}x^2 - 7x\right] + C$]

Ex.2: Suppose a firm has a marginal cost

$$MC = a + bQ + dQ^2$$

Find the equation of its total cost curve.

[Ans: $C(Q) = aQ + \frac{bQ^2}{2} + \frac{dQ^3}{3} + e$. e is the constant of integration. It is the value of $C(Q)$ when $Q = 0$, i.e. the fixed cost of production.]

> If a value of $f(x_i)$ [initial boundary condition] is given,
> the constant of integration (C) can be evaluated to obtain a specific equation.

Ex.3: If $f'(x) = 2x$ and $f(1) = 4$, find $f(x)$. [ANS: $x^2 + 3$]

Ex.4: Find the function $f(x)$ whose tangent has slope $\sqrt{x\sqrt{x\sqrt{x}}}$ for each value of x and whose graph passes through the point (1,1). [ANS: $\frac{8}{15}x^{\frac{15}{8}} + \frac{7}{15}$]

Ex.5: For a firm, the marginal cost is $MC = 3Q + 4$. If the fixed cost are 40, find the total cost function *C*(*Q*). [ANS: $C = \frac{3}{2}Q^2 + 4Q + 40$]

If there are *n* numbers of constants of integration, *n* numbers of initial boundary conditions $f(x_i)$ are required to evaluate all constants of integration.

Ex.6: If $y'' = x^2 + x + 1$, find $y(x)$ for [ANS: $\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} + C_1 x + C_2$]

(i) when $y(0) = 0$ and $y(1) = \frac{7}{4}$ [ANS: $\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} + x$]

(ii) when $y(0) = 0$ and $y'(0) = 0$ [ANS: $\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2}$]

NOTE that if $y'(a)$ and $y'(b)$ are given, these are not enough because C_2 cannot be determined.

Ex.7: Given the marginal-revenue function $MR = 2000 - 20Q - 3Q^2$. Find the demand function. [ANS: $2000 - 10Q - Q^2$]

Hint: If there is no product sold $(Q = 0)$ then the revenue is zero.

6.3 Power Rule for Integration and Integration by Substitution.

$$\frac{d}{dx}\left(\frac{[u(x)]^{n+1}}{n+1}\right) = [u(x)]^n \cdot u'(x)$$

$$\therefore \quad \int [u(x)]^n \cdot u' dx = \frac{[u(x)]^{n+1}}{n+1} + C$$

But $\quad u'(x) = \frac{du}{dx} \Rightarrow u'(x)dx = du$

$$\boxed{\therefore \quad \int [u(x)]^n du = \frac{u^{n+1}}{n+1} + C}$$

Power Rule

> If u is differentiable and a function of x, then $\int u^n du = \frac{u^{n+1}}{n+1} + C$ if $n \neq -1$.

Ex.8: Find $\int (x+5)^5 dx$.

Integration by substitution, try $\quad u = x + 5 \Rightarrow du = 1 \cdot dx$

$$\int (x+5)^5 dx = \int u^5 du = \frac{u^6}{6} + C$$

Substitute u gives $\quad \int (x+5)^5 dx = \frac{(x+5)^6}{6} + C$

Note that you **MUST** give the answer explicitly in term or x **NOT** u.

Ex.9: Find $\int \sqrt[3]{6x} dx$. (**Hint:** u=6x) [ANS: $\frac{(6x)^{\frac{4}{3}}}{8} + C$]

Ex.10: Show that $\int x^2\left(3x^3+7\right)^3 dx = \frac{(3x^3+7)^4}{36}+C$ [**Hint:** $u = 3x^3+7$]

6.4 More Integration by Substitution

Let u be a function of x,

$$\int e^u du = e^u + C$$

$$\int \frac{1}{u} du = \ln|u| + C$$

Ex.11: Show that $\int e^{ax} dx = \frac{1}{a}e^{ax} + C$. [**Hint:** $u = ax$]

Ex.12: Show that $\int a^x dx = \frac{1}{\ln a}a^x + C$.

Ex.13: Show that $\int \left((2t+1)e^{t^2+t}dt\right) = e^{t^2+t} + C$. [**Hint:** $u = t^2+t$]

Ex.14: Show that $\int \frac{7t}{5t^2-6}dt = \frac{7}{10}\ln\left|5t^2-6\right| + C$.

Fill in the table by specifying the substitution you may choose for each integral.

Integral	**Substitution *u***
1. $\int (3x+4)^{\frac{5}{2}} dx$	
2. $\int \left(\frac{4}{3-x}\right) dx$	
3. $\int te^{2-t^2} dt$	
4. $\int t\left(2+t^2\right)^3 dt$	

5. $\int \frac{3}{(2x-5)^4}\,dx$	
6. $\int x^2 e^{-x^3}\,dx$	
7. $\int \frac{e^t}{e^t+1}\,dt$	
8. $\int \frac{t+3}{\sqrt[3]{t^2+6t+5}}\,dt$	

6.4.1 Integration Hints

- Try to arrange the integrand into a familiar integration forms (seen formulas). If you can't find the right form, try some more! There is no shortcut! **ONLY PRACTICE!**

- When integrating fraction forms, sometimes a preliminary long division is needed to get familiar integration forms.

Ex.15: Show that $\int\left(\frac{x+3}{x+6}\right)dx = x - 3\ln|x+6| + C$.

Ex.16: Show that $\int\left(\frac{x^3+x^2-x-3}{x^2-3}\right)dx = \frac{x^2}{2} + x + \ln|x^2-3| + C$.

Ex.17: Show that $\int\left(\frac{9x^2+5}{3x}\right)dx = \frac{3}{2}x^2 + \frac{5}{3}\ln|x| + C$.

Ex.18: Show that $\int\left(\frac{6x^2-11x+5}{3x-1}\right)dx = x^2 - 3x + \frac{2}{3}\ln|3x-1| + C$.

Ex.19: Show that $\int 2x(x^2+3)^5\,dx = \frac{(x^2+3)^6}{6} + C$.

Ex.20: Show that $\int x^3 e^{4x^4}\,dx = \frac{1}{16}e^{4x^4} + C$.

Ex.21: $y' = (3-2x)^2$, $y(0)=1$, $y(x)=?$. [ANS: $-\frac{1}{6}(3-2x)^3 + \frac{11}{2}$]

Ex.22: $y'' = \frac{1}{x^2}$, $y'(-1)=1$, $y(1)=0$, $y(x)=?$. [ANS: $-\ln|x|$]

6.5 Summation (Revisions)

$$\sum_{i=0}^{n} i = 0+1+2+.....+n = \sum_{k=0}^{n} k$$

$\sum$ = notation for summation called ‘sigma’.

i, k = index of summation (a ‘dummy’ symbol)

0, n = limits of summation

If n = 5, $\sum_{i=0}^{5} i = 0+1+2+3+4+5$

Ex.23: Show that $\sum_{i=1}^{5} 3i^2 = 3\sum_{i=0}^{5} i^2$.

ANS:

$$\sum_{i=1}^{5} 3i^2 = 3(1)^2 + 3(2)^2 + 3(3)^2 + 3(4)^2 + 3(5)^2$$

$$= 3(1^2 + 2^2 + 3^2 + 4^2 + 5^2)$$

$$= 3\sum_{i=0}^{5} i^2$$

$$\boxed{\sum_{i=0}^{n} Cx_i = C\sum_{i=0}^{n} x_i}$$

Ex.24: If $\sum_{k=1}^{n} k = 1+2+3+....+n$, show that $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$ is the general form in term of *n*.

ANS: Let $\quad S = \sum_{k=1}^{n} k = 1+2+3...+n \qquad (1)$

Reverse $\quad S = n+(n-1)+(n-2)...+1 \qquad (2)$

(1)+(2) $\quad 2S = (n+1)+(n-1+2)+(n-2+3)+...+(n+1)$

$$2S = (n+1)+(n+1)+(n+1)+...+(n+1)$$

$$2S = n\cdot(n+1)$$

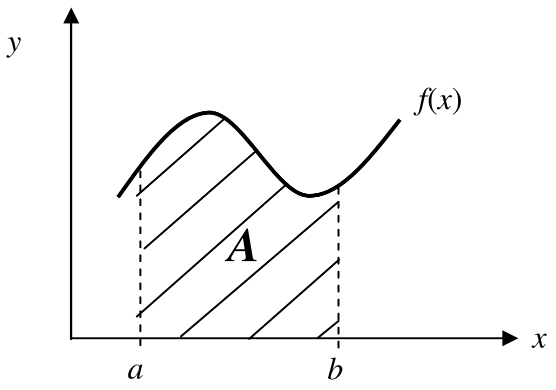
$$S = \frac{n\cdot(n+1)}{2} = \sum_{k=1}^{n} k$$

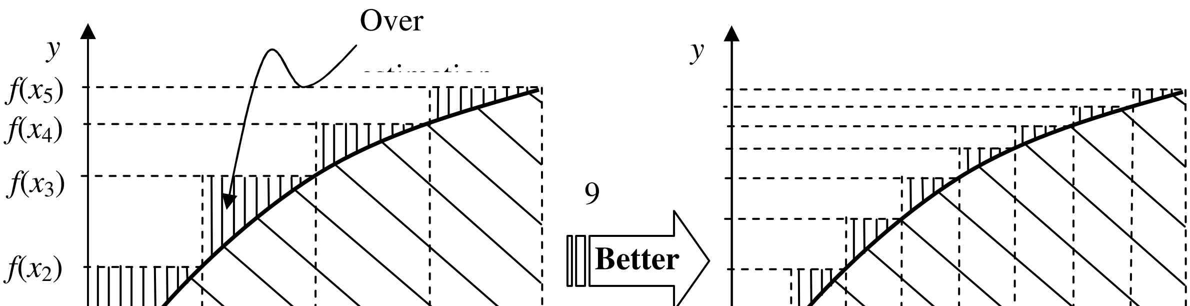
Ex.25: Find $\sum_{i=1}^{5} i$ and $\sum_{i=1}^{100} i$. [ANS: 15, 5050]

6.6 Definite Integral as an Area under a Curve

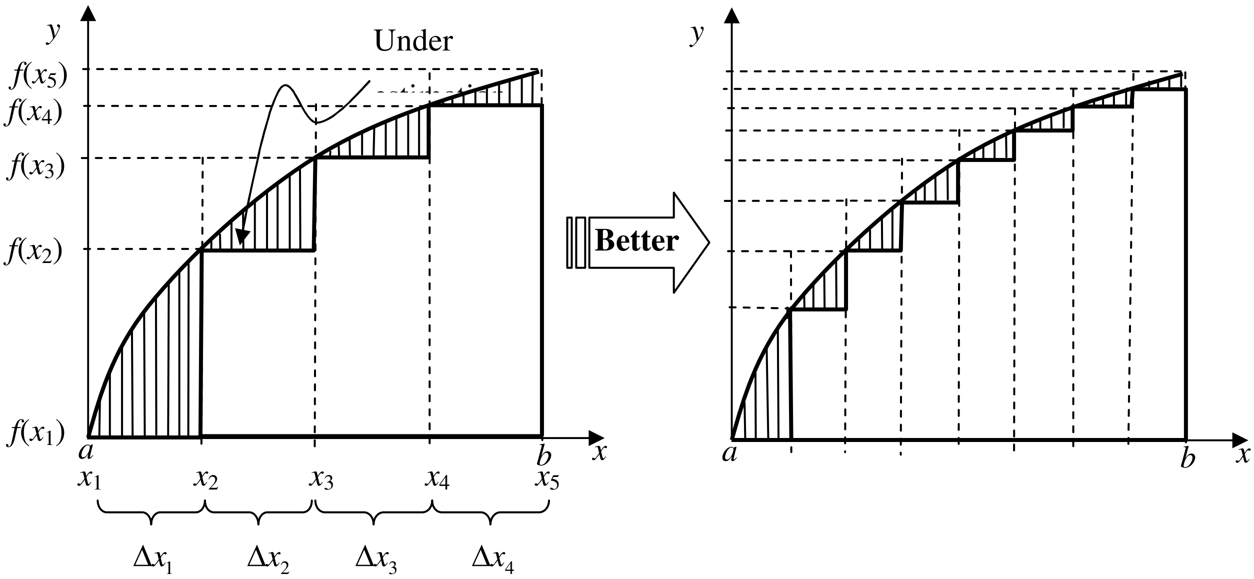
Area under a curve can be expressed as a limit of a sum of terms called a **definite integral.**

$$A = \int_a^b f(x)dx = \text{Area under a curve between } x = [a, b]$$

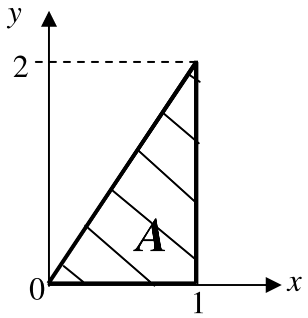




$$A^* = f(x_2)\Delta x_1 + f(x_3)\Delta x_2 + f(x_4)\Delta x_3 + f(x_5)\Delta x_4 = \sum_{i=1}^{4} f(x_{i+1})\Delta x_i$$

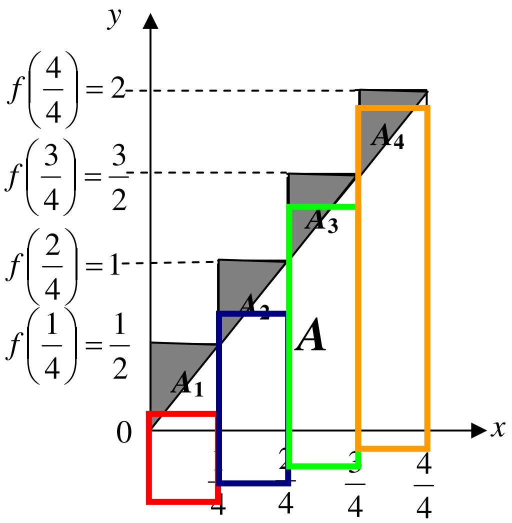


$$A^* = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + f(x_4)\Delta x_4 = \sum_{i=1}^{4} f(x_i)\Delta x_i$$



$f(x) = 2x$

$A = \frac{1}{2} \cdot 1 \cdot 2 = 1$



$A_1 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

$A_2 = \frac{1}{4} \times 1 = \frac{1}{4}$

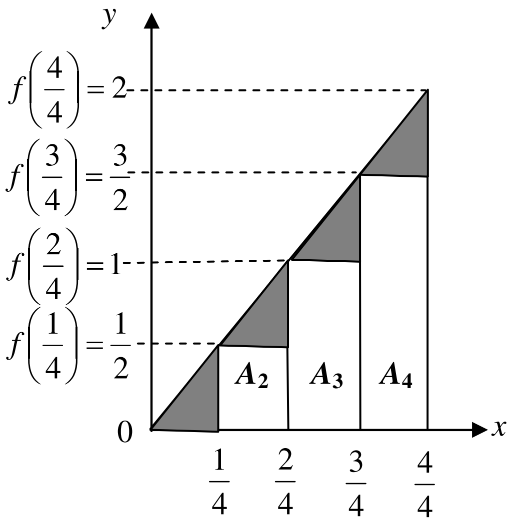
$A_3 = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$

$A_4 = \frac{1}{4} \times 2 = \frac{1}{2}$

Estimated area (Over estimate)

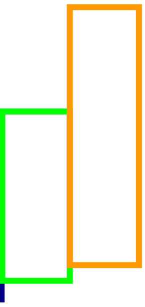
$$\overline{S}_4 = A_1 + A_2 + A_3 + A_4 = \left(\frac{1+2+3+4}{8}\right) = \frac{5}{4}$$

Over estimate by $= \frac{5}{4} - 1 = \frac{1}{4}$



$A_1 = 0$

$A_2 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$



$$A_3 = \frac{1}{4} \times 1 = \frac{1}{4}$$

$$A_4 = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$$

Estimated area (Under estimate)

$$\underline{S}_4 = A_1 + A_2 + A_3 + A_4 = \left(\frac{0+1+2+3}{8}\right) = \frac{3}{4}$$

Under estimate by $= 1 - \frac{3}{4} = \frac{1}{4}$

$$y = f(x) = 2x, \qquad x = [0,1], \qquad \Delta x = \frac{1}{4}$$

$$\overline{S}_4 = \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f\left(\frac{4}{4}\right)$$

$$\overline{S}_4 = \frac{1}{4}\left[2\left(\frac{1}{4}\right) + 2\left(\frac{2}{4}\right) + 2\left(\frac{3}{4}\right) + 2\left(\frac{4}{4}\right)\right]$$

$$\overline{S}_4 = \frac{5}{4} > 1 \quad \text{(Exact area} = 1)$$

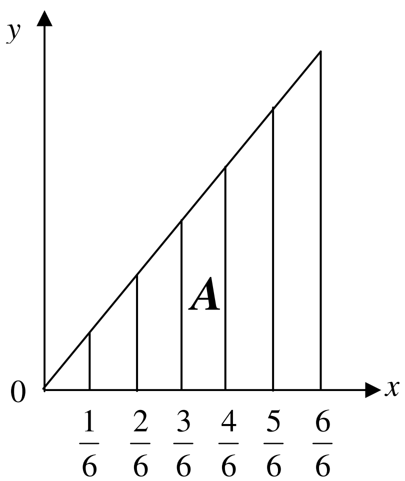
$$\underline{S}_4 = \frac{1}{4} f(0) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right)$$

$$\underline{S}_4 = \frac{1}{4}\left[2(0) + 2\left(\frac{1}{4}\right) + 2\left(\frac{2}{4}\right) + 2\left(\frac{3}{4}\right)\right]$$

$$\underline{S}_4 = \frac{3}{4} < 1 \quad \text{(Exact area} = 1)$$

$$\overline{S}_4 = \sum_{i=1}^{4} f(x_i)\Delta x$$

$$\underline{S}_4 = \sum_{i=1}^{3} f(x_i)\Delta x$$



$$f(x) = 2x$$

$$x = [0,1]$$

$$\Delta x = \frac{1}{6}$$

$$\overline{S}_6 = \sum_{i=1}^{6} f(x_i)\Delta x$$

$$\underline{S}_6 = \sum_{i=1}^{5} f(x_i)\Delta x$$

$$\overline{S}_6 = \frac{1}{6} f\left(\frac{1}{6}\right) + \frac{1}{6} f\left(\frac{2}{6}\right) + \frac{1}{6} f\left(\frac{3}{6}\right) + \frac{1}{6} f\left(\frac{4}{6}\right) + \frac{1}{6} f\left(\frac{5}{6}\right) + \frac{1}{6} f\left(\frac{6}{6}\right)$$

$$\overline{S}_6 = \frac{1}{6}\left[2\left(\frac{1}{6}\right) + 2\left(\frac{2}{6}\right) + 2\left(\frac{3}{6}\right) + 2\left(\frac{4}{6}\right) + 2\left(\frac{5}{6}\right) + 2\left(\frac{6}{6}\right)\right]$$

$\overline{S}_6 = \frac{1}{18}[1+2+3+4+5+6] = \frac{21}{18} = \frac{7}{6}$ Over estimate by $= \frac{7}{6} - 1 = \frac{1}{6}$

$\underline{S}_6 = \frac{1}{6} f(0) + \frac{1}{6} f\left(\frac{1}{6}\right) + \frac{1}{6} f\left(\frac{2}{6}\right) + \frac{1}{6} f\left(\frac{3}{6}\right) + \frac{1}{6} f\left(\frac{4}{6}\right) + \frac{1}{6} f\left(\frac{5}{6}\right)$

$\underline{S}_6 = \frac{1}{6}\left[2(0) + 2\left(\frac{1}{6}\right) + 2\left(\frac{2}{6}\right) + 2\left(\frac{3}{6}\right) + 2\left(\frac{4}{6}\right) + 2\left(\frac{5}{6}\right)\right]$

$\underline{S}_6 = \frac{1}{18}[0+1+2+3+4+5] = \frac{15}{18} = \frac{5}{6}$ Under estimate by $= 1 - \frac{5}{6} = \frac{1}{6}$

The estimated result is getting closer as the interval (Δx) gets **SMALLER!**

General Term $f(x) = 2x$ Divide $x = [0,1]$ into n subintervals. Hence, $\Delta x = \frac{1}{n}$

$\overline{S}_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$\overline{S}_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \ldots + \frac{1}{n} f\left(\frac{n}{n}\right)$

$\overline{S}_n = \frac{1}{n}\left[2\left(\frac{1}{n}\right) + 2\left(\frac{2}{n}\right) + 2\left(\frac{3}{n}\right) + \ldots + 2\left(\frac{n}{n}\right)\right]$

$\overline{S}_n = \frac{2}{n^2} \cdot [1+2+3+\ldots+n]$

$\overline{S}_n = \frac{2}{n^2} \cdot \left[\frac{n(n+1)}{2}\right] = \frac{n+1}{n}$ Note: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$\underline{S}_n = \sum_{i=1}^{n-1} f(x_i)\Delta x$

$\underline{S}_n = \frac{1}{n} f\left(\frac{0}{n}\right) + \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \ldots + \frac{1}{n} f\left(\frac{n-1}{n}\right)$

$\underline{S}_n = \frac{1}{n}\left[2\left(\frac{0}{n}\right) + 2\left(\frac{1}{n}\right) + 2\left(\frac{2}{n}\right) + \ldots + 2\left(\frac{n-1}{n}\right)\right]$

$\underline{S}_n = \frac{2}{n^2} \cdot [0+1+2+\ldots+(n-1)]$

$\underline{S}_n = \frac{2}{n^2} \cdot \left[\frac{(n-1)n}{2}\right] = \frac{n-1}{n}$ Note: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

As subinterval is getting smaller ($\Delta x \to 0$) ➔ n is larger ➔ $\overline{S}_n, \underline{S}_n$ closer to the exact area A.

$$\overline{S}_n = \frac{n+1}{n} \qquad \underline{S}_n = \frac{n-1}{n}$$

Question: For $\overline{S}_n = \frac{n+1}{n}$ and $\underline{S}_n = \frac{n-1}{n}$, what happen if $n \to \infty$?

$$\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

$$\lim_{n\to\infty} \underline{S}_n = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1$$

$$\therefore \quad \lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \underline{S}_n = 1$$

As $n \to \infty$, $\overline{S}_n \to A$ and $\underline{S}_n \to A$ (A = 1, exact area)

$$\underline{S}_n \le A \le \overline{S}_n$$

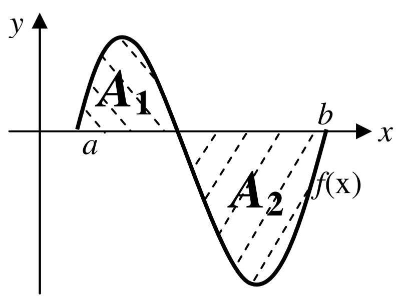
Common limit of $\underline{S}_n$ and $\overline{S}_n$ ➔ Definite integral of $f(x)$

$$\therefore A = \int_0^1 2x\,dx = 1$$

The **common** limit of $\underline{S}_n$ and $\overline{S}_n$ as $n \to \infty$, if it exists, is called the **definite integral** of $f(x)$ over (a,b).

$$A = \int_a^b f(x)dx$$

For example,



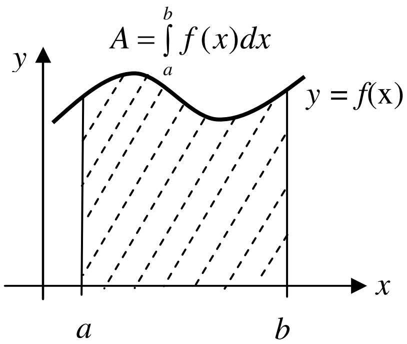
$$\int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1^2 - 0^2 = 1 = \lim_{n\to\infty} \underline{S}_n = \lim_{n\to\infty} \overline{S}_n$$

NOTE: Definite integral **DOES NOT** always represents an area.

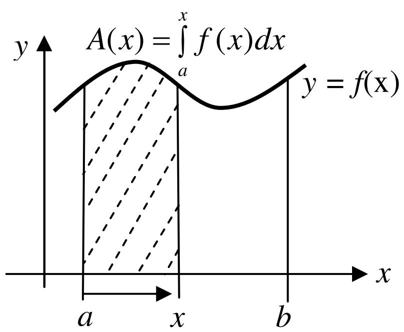
$\int_a^b f(x)dx = A_1 - A_2 =$ **NEGATIVE.** Area can't be negative!

6.7 Fundamental Theorem of Integration Calculus

- To informally develop the fundamental theorem of integral calculus and to use it to complete definite integrals.
- To obtain a change in function values when the rate of change of the function is known.

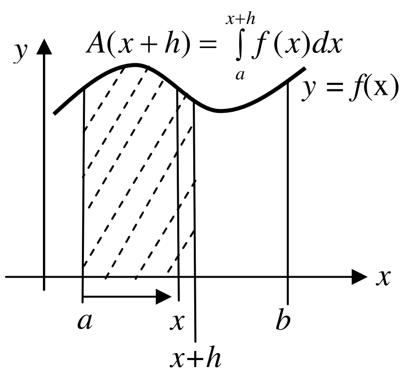


A = Area between the curve $y = f(x)$, $x = a$, $x = b$, and the x axis.



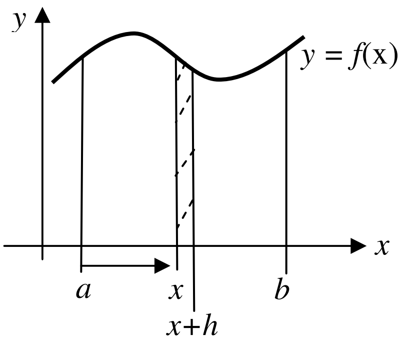
A = Area function

$A(x)$ = Area between the curve $y = f(x)$, $x = x$, $x = b$, and the x axis.



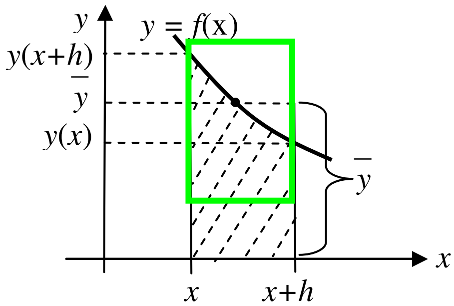
Increase x by h unit

$A(x+h)$ = Area between $y = f(x)$, $x = (x+h)$, $x = b$, and the x axis.



The different in area

$$\Delta A = A(x+h) - A(x)$$



As $h \to 0$, area [hatched box] is approximately rectangle.

Estimate to rectangle with height $\bar{y}$

$$\Delta A = A(x+h) - A(x) = \bar{y} \cdot h$$

$$\bar{y} = \frac{A(x+h) - A(x)}{h}$$

$$\bar{y} = \frac{A(x+h) - A(x)}{h}$$

$h \to 0$ and $\bar{y} \to f(x)$

$$\therefore \quad \lim_{h \to 0}\left[\frac{A(x+h) - A(x)}{h}\right] = f(x)$$

This is the definition of a derivative of *A*.

$$\therefore \quad A'(x) = f(x)$$

A **derivative** of an area *A* is *f*(*x*) the function itself.

A is an **anti-derivative** of *f*(*x*).

$$\therefore \quad \int f(x)dx = A(x) + C_1 \qquad (1)$$

The integral of a function *f*(*x*) is the area function *A*(*x*). Anti-derivative of *f*(*x*) is *F*(*x*).

$$\therefore \quad \int f(x)dx = F(x) + C_2 \qquad (2)$$

(1) = (2), $\qquad A(x) + C_1 = F(x) + C_2$

$$A(x) = F(x) + C \qquad (3)$$

$$A(0) = 0 = F(a) + C \Rightarrow C = -F(a)$$

(3) becomes

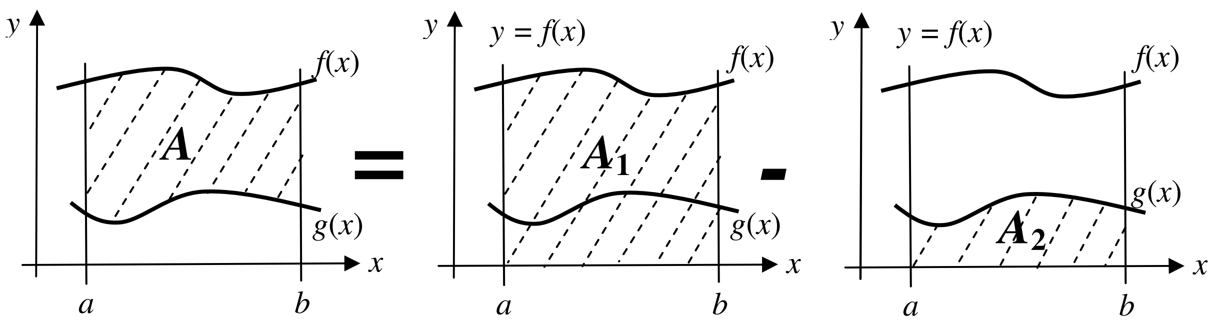
$$A(x) = F(x) - F(a) \qquad (4)$$

If $x = b$, (4) becomes

$$A(x) = F(b) - F(a) = \int_a^b f(x)dx$$

Hence, if *f* is continuous on interval $[a,b]$, $F(x)$ is any anti-derivative of *f*(*x*) on the interval then

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$



6.7.1 Area between Curves

Find area bounded by *f*(*x*), *g*(*x*), *x* = *a*, and *x* = *b*.

$$A = A_1 - A_2$$

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$A = \int_a^b \left[f(x) - g(x)\right]dx$$

$$\boxed{\therefore \quad \int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx}$$

Similarly others properties of definite integral can be derived from area between curves.

6.7.2 Properties of Definite Integral

1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2) $\int_a^a f(x)dx = 0$

3) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (k is a constant)

4) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

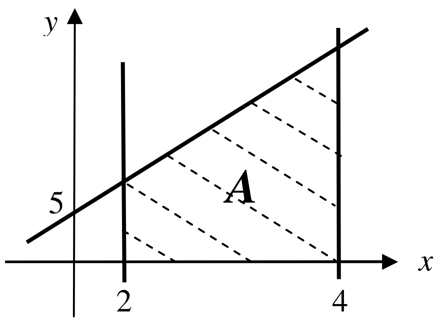
5) $\int_a^b f(x)dx = -\int_b^a f(t)dt$ Any variable will give the same result.

6) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

Ex.26: Use a definite integral to find the area of the region bounded by the given curve, the x axis, and the given lines.

(***Note: We need to sketch the curve to make sure that there will be no negative area!***)

(i) $y = x + 5$, $x = 2$, and $x = 4$



$$A = \int_2^4 (x+5)dx = \left[\frac{x^2}{2} + 5x\right]_2^4$$

$$A = \left(\frac{4^2}{2} + 5(4)\right) - \left(\frac{2^2}{2} + 5(2)\right)$$

$$A = 28 - 12 = 16$$

(ii) $y = e^x$, $x = 1$, and $x = 3$ [Ans: $e^3 - e$]

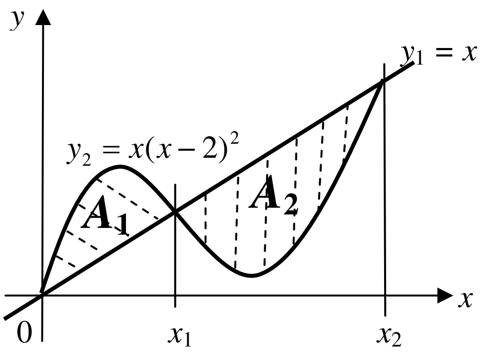
(iii) $y = x^2 - 2x$, $x = 1$, and $x = 3$ [Ans: 2]

(iv) $y = x^3$, $x = -2$, and $x = 4$ [ANS: 68]

(v) $y = \dfrac{1}{x}$, $x = 1$, and $x = e$ [ANS: 1]

(vi) $y = \dfrac{1}{x}$, $x = 1$, and $x = e^2$ [ANS: 2]

Ex.27: Express the area in terms of integrals



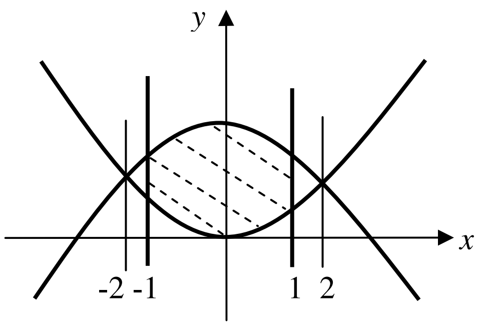
Find intersection points

$$x(x-2)^2 = x$$

$A =$

Ex.28: Find the area of the region bounded by the graphs of the given equations. Be sure to find any needed points of intersection.

(i) $y = 8 - x^2$, $y = x^2$, $x = -1$, and $x = 1$ [Ans: 44/3]



Intersection: $8 - x^2 = x^2$

$$2x^2 = 8 \Rightarrow x = \pm 2$$

$$A = \int_{-1}^{1} \left[\left(8 - x^2\right) - x^2\right] dx = \frac{44}{3}$$

(ii) $y = 2 - x^2$, $y = x$ [Ans: $\int_{-2}^{1} \left[\left(2 - x^2\right) - x\right] dx = \frac{9}{2}$]

(iii) $y = x^2 + 2$, $y = 8$ [Ans: $\int_{-\sqrt{6}}^{\sqrt{6}} \left[8 - \left(2 - x^2\right)\right] dx = 8\sqrt{6}$]

6.7.3 The Average Value of a Function

Let $f(x)$ be a function that is continuous on the interval $a \le x \le b$. Then the ***average value*** *V* of $f(x)$ over $a \le x \le b$ is given by the definite intergral

$$V = \frac{1}{b-a}\int_a^b f(x)dx$$

Ex.29: A manufacture determines that t moths after introducing a new product, the company's sales will be $S(t)$ thousand dollars, where

$$S(t) = \frac{750t}{\sqrt{4t^2+25}}$$

What are the average monthly sales of the company over **the first 6 months** after the introduction of the new product? [Ans: 250]

6.8 Definite Integral (With numerical limit, the answer is real number.)

If $F(x)$ is any anti-derivative of the function $f(x)$, the 'definite' integral of $f(x)$ between *a* and *b*, $(a \le x \le b)$, is

$$\int_a^b f(x)dx = F(b) - F(a) = \left[F(x)\right]_a^b \qquad a < b$$

a is the lower limit of integration and *b* is the upper limit of integration. **Definite** integral will give a **definite** numerical value, **NO *x* or *C*.** *C* will cancel itself out during the calculation.

Ex.30: Show that $\int_1^5 3x^2 dx = 124$ and $\int_0^2 (4 - x^2)\, dx = \frac{16}{3}$.

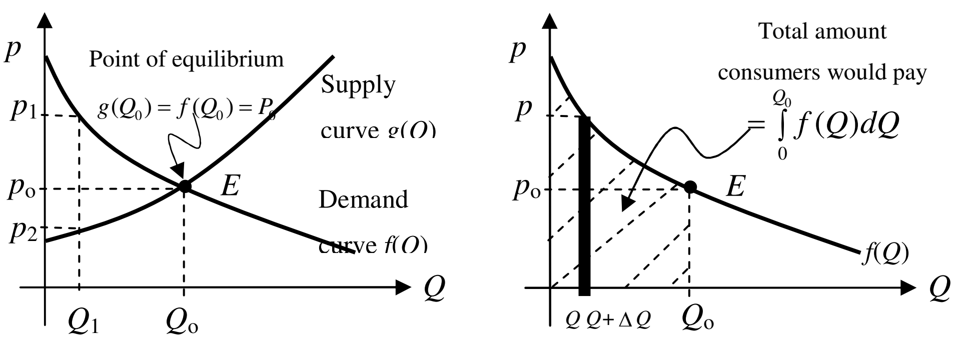
When using integration by substitution for a definite integral, we need to **substitute the limit** as well or **substitute before taking limits**.

Ex. 31: Show that $\int_1^2 (2x^3 - 1)^2 (6x^2)\, dx = \frac{3374}{3}$.

Ex. 32: Show that $\int_9^{64} \left(\frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}} \right) dx = \frac{76}{3}$.

6.9 Consumers' and Producers' Surplus

Economists are interested in studying how consumers' and producers' welfare are influenced by changes in the economic parameters. A common measure of such welfare changes is the consumers' and producers' surplus.



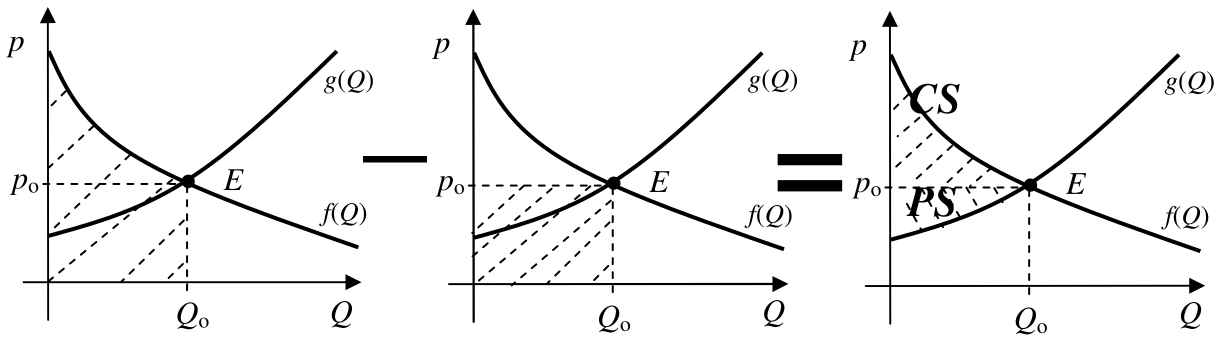
p = price **per unit (example of unit is bahts/unit)** Q = Quntity buy or sell

- Supply curve $[g(Q)]$ indicates the price p per unit at which the **producers** will **sell** (or **supply**) Q units.
- Demand curve $[f(Q)]$ indicates the price p per unit at which the **consumers** will **purchase** (or **demand**) Q units.

- **Point of equilibrium** (E) indicates the consumers to purchase precisely the same amount (Q_0) that the producers are willing to offer at the price per unit (p_0).

  Demand and supply curves are intercepted. $p_0 = q(Q_0) = f(Q_0)$

- If **the market is at equilibrium** $(p = p_0)$, there are consumers who are willing to **pay more** than p_0. For example, at the price of p_1, consumers would buy Q_1 unit. These consumers **benefiting** from the lower price p_2.
- $p\Delta Q$ is the total amount of money that customers would spend by buying ΔQ unit if the price per unit was p. In fact, the price is at the equilibrium p_0 so the customers only buy $p_0\Delta Q$. Thus, the customers ***save*** or ***benefit*** $(p - p_0)\Delta Q$.
- Area under the curve is the amount of money to sell or buy the product.



$\int_0^{Q_0} f(Q)dQ$ - $\int_0^{Q_0} P_0 dQ$ = $\int_0^{Q_0}[f(Q) - P_0]dQ = CS$

- For those who are willing to pay for commodity at price p_0 or higher, the total amount they are willing to pay is the total area $\int_0^{Q_0} f(Q)dQ$. In fact, consumers only pay $\int_0^{Q_0} P_0 dQ$. Thus, the customers ***save***

$$CS = \int_0^{Q_0}\left[f(Q) - P_0\right]dQ.$$

- **Consumers' Surplus [*CS*]** is the total amount ***save*** by all the customers buying he commodity at a price lower than what they are maximally are willing to pay.
- Some producers are also benefiting from the equilibrium price, since they are willing to supply at prices ***lower*** than p_0. Similarly,

$$PS = \int_0^{Q_0}\left[P_0 - f(Q)\right]dQ$$

- **Producers' Surplus [*PS*]** is the total benefit of all the producers who obtain a price higher than the price they are willing to sell it for.

Ex.33: A demand function is $q = 10000e^{-0.02p}$ unit. At what price will revenue be maximised? [Ans: 50 units]

Ex.34: A demand function is $p = f(q) = 100 - 0.05q$ unit. A supply function is $p = g(q) = 10 + 0.1q$ bahts/unit. Determine consumers' and producers' surplus under market equilibrium.

[Ans: 600 units, 70 bahts/unit, 9000 bahts, 18000 bahts]

Solution: At equilibrium point (q_0, p_0), the demand and supply functions are intercepted.

$$100 - 0.05q = 10 + 0.1q$$
$$0.15q = 90$$
$$q_0 = 600 \text{ units}$$
$$p_0 = 10 + 0.1q_0 = 10 + 0.1(600)$$
$$p_0 = 70 \text{ bahts/unit}$$

$$CS = \int_0^{q_0} [f(q) - p_0]dq$$
$$= \int_0^{600} [\{100 - 0.05q\} - 70]dq$$
$$= \left[30q - \frac{0.05q^2}{2}\right]_0^{600}$$
$$= \left[30(600) - \frac{0.05(600)^2}{2}\right] - 0$$
$$= 9{,}000 \text{ bahts}$$

$$PS = \int_0^{q_0} [p_0 - f(q)]dq$$
$$= \int_0^{600} [70 - \{10 + 0.1q\}]dq$$
$$= \left[60q - \frac{0.1q^2}{2}\right]_0^{600}$$
$$= \left[60(600) - \frac{0.1(600)^2}{2}\right] - 0$$
$$= 18{,}000 \text{ bahts}$$

Ex.35: A demand function is $p = \dfrac{50}{q+5}$ unit. A supply function is $p = \dfrac{q}{10} + 4.5$ bahts/unit.

Determine consumers' and producers' surplus under market equilibrium.

[Ans: 5 units, 5 bahts/unit, 50ln2-25 bahts, 1.25 bahts]

Ex.36: A demand function is $p = 2^{11-q}$. A supply function is $p = 2^{q+1}$. Determine consumers' surplus under market equilibrium.

[Ans: 5 unit, 64 bahts/unit, $\left(\dfrac{2^{11}}{\ln 2} - \dfrac{64}{\ln 2} - 320\right)$ bahts]