

Solution: Assignment 6

1. Consider the relation f defined from X to Y , $X, Y \subseteq \mathbb{R}$,

$$f = \left\{ (x, y) \in X \times Y \mid x^3 + yx^2 - y = 0 \right\}.$$

Note: It might be useful to write y as a function of x , i.e. $y = f(x)$, before answering some questions.

- (1.1) Find the domain of f .
- (1.2) Find x -intercepts and y -intercepts (if any).
- (1.3) Determine the symmetry of f .
- (1.4) Find the horizontal and vertical asymptotes for f (if any).
- (1.5) Show that $y = -x$ is a slant asymptote of f .
- (1.6) Find the critical number of f . Determine the intervals on which f is increasing and decreasing. Determine the extrema (maximum and minimum) of f .
- (1.7) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).
- (1.8) Sketch the curve of f .

Answer: Notice that we can rearrange as $y = \frac{x^3}{1-x^2}$. So we can write this as

$$f(x) = \frac{x^3}{1-x^2}$$

- 1.1 Since $1 - x^2$ is the denominator, so we must have $1 - x^2 \neq 0$. The domain of f (which consists of possible values of x) is $\mathbb{R} - \{1, -1\}$. ■
- 1.2 x -intercepts: setting $y = 0$ gives $0 = \frac{x^3}{1-x^2}$, which occurs if and only if

$$x^3 = 0 \quad \Leftrightarrow \quad x = 0.$$

So x -intercepts is $(0, 0)$.

To find y -intercepts, set $x = 0$, which implies $y = 0$. So $(0, 0)$ is also a y -intercept of f . ■

- 1.3 Determine the symmetry of f .
Notice that f is a function and can be written as $f(x) = \frac{x^3}{1-x^2}$. Hence, it is not symmetric about the x -axis. We will look at the symmetry by using $f(-x)$.

$$f(-x) = \frac{(-x)^3}{1-(-x)^2} = -\frac{x^3}{1-x^2}.$$

Since $f(-x) = -f(x)$, f is an odd function and it is symmetric about the origin.

Note that $f(-x) \neq f(x)$, it is not symmetric about the y -axis. ■

- 1.4 Find the horizontal and vertical asymptotes for f (if any).

Horizontal asymptotes:

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^3}{1-x^2} = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^3}{1-x^2} = -\infty$$

So, there is no horizontal asymptote. ■

Vertical asymptotes: Notice that the function is not defined at $x = \pm 1$.

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} \frac{x^3}{1-x^2} = \infty & \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} \frac{x^3}{1-x^2} = -\infty \\ \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} \frac{x^3}{1-x^2} = -\infty & \lim_{x \rightarrow -1^+} f(x) &= \lim_{x \rightarrow -1^+} \frac{x^3}{1-x^2} = \infty \end{aligned}$$

So, the vertical asymptotes are $x = -1$ and $x = 1$. ■

1.5 To show that $y = -x$ is a slant asymptote, we must have $\lim_{x \rightarrow \infty} (f(x) - (-x)) = 0$:

$$\lim_{x \rightarrow \infty} (f(x) - (-x)) = \lim_{x \rightarrow \infty} \frac{x^3}{1-x^2} + x = \lim_{x \rightarrow \infty} \frac{x^3 + x - x^3}{1-x^2} = \lim_{x \rightarrow \infty} \frac{x}{1-x^2} = 0$$

That is, $y = -x$ is a slant asymptote by the limit definition. ■

Alternative answer: Suppose the slant asymptote is in the form of $y = mx + b$. We will determine this asymptote by finding the values of m and b such that

$$\lim_{x \rightarrow \infty} (f(x) - [mx + b]) = 0.$$

From $f(x) = \frac{x^3}{1-x^2}$,

$$\begin{aligned} \lim_{x \rightarrow \infty} (f(x) - [mx + b]) &= \lim_{x \rightarrow \infty} \left(\frac{x^3}{1-x^2} - [mx + b] \right) = \lim_{x \rightarrow \infty} \frac{x^3 - (mx + b)(1-x^2)}{1-x^2} \\ &= \lim_{x \rightarrow \infty} \frac{x^3 - (mx - mx^3) - (b - bx^2)}{1-x^2} \\ &= \lim_{x \rightarrow \infty} \frac{(1+m)x^3 - bx^2 - mx - b}{1-x^2}, \end{aligned}$$

which will become zero only when the degree of x in the numerator is **smaller** than the degree of x in the denominator. Hence the coefficients of x^3 and x^2 in the numerator have to be zero:

$$1 + m = 0 \quad \text{and} \quad b = 0$$

or $\boxed{m = -1 \text{ and } b = 0}$. By substituting $m = -1, b = 0$ in $y = mx + b$, we have that $y = -x$ is a slant asymptote of the curve $y = f(x)$. ■

1.6 Find the critical number of f .

$$f'(x) = \frac{3(1-x^2)x^2 - (-2x)x^3}{(1-x^2)^2} = \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2} = \frac{3x^2 - x^4}{(1-x^2)^2} = \frac{x^2(3-x^2)}{(1-x^2)^2}$$

The critical number x occurs when $f'(x) = 0$ or $f'(x)$ does not exist.

$$f'(x) = 0 \Leftrightarrow 3 - x^2 = 0, x^2 = 0 \Leftrightarrow x = -\sqrt{3}, 0, \sqrt{3}.$$

Note that $f'(x)$ does not exist when $x = \pm 1$, which is not in the domain, so ± 1 is not a critical number. Hence, the critical numbers are $x = -\sqrt{3}, 0, \sqrt{3}$. ■

◆ Determine the intervals on which f is increasing and decreasing. Determine the relative extrema (maximum and minimum) of f .

From

$$f'(x) = \frac{x^2(3-x^2)}{(1-x^2)^2} = \frac{x^2(\sqrt{3}+x)(\sqrt{3}-x)}{(1-x^2)^2}$$

- Decreasing: $f'(x) < 0$ when $\frac{x^2(\sqrt{3}+x)(\sqrt{3}-x)}{(1-x^2)^2} < 0$
- Increasing: $f'(x) > 0$ when $\frac{x^2(\sqrt{3}+x)(\sqrt{3}-x)}{(1-x^2)^2} > 0$

We will consider 6 intervals from $x = -\sqrt{3}, -1, 0, 1, \sqrt{3}$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
Sign of $f'(x) = \frac{x^2(\sqrt{3}+x)(\sqrt{3}-x)}{(1-x^2)^2}$ $-(x+\sqrt{3})(x-\sqrt{3})$	- $\frac{(+)(-)(+)}{(+)}$	+ $\frac{(+)(+)(+)}{(+)}$	+ $\frac{(+)(+)(+)}{(+)}$	+ $\frac{(+)(+)(+)}{(+)}$	+ $\frac{(+)(+)(+)}{(+)}$	- $\frac{(+)(+)(-)}{(+)}$

I.e., $f'(x) < 0$ when $x < -\sqrt{3}$ and $x > \sqrt{3}$. Also, $f'(x) > 0$ for $x \in (-\sqrt{3}, \sqrt{3})$ or $x \in (-\sqrt{3}, -1) \cup (-1, 1) \cup (1, \sqrt{3})$ (note: $x = \pm 1$ are not in the domain)

- Decreasing interval: $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$
- Increasing interval: $(-\sqrt{3}, -1) \cup (-1, 1) \cup (1, \sqrt{3})$.

Therefore, from the *First Derivative Test*,

- relative maximum occurs at $x = \sqrt{3}$ (the sign of $f'(x)$ changes from “+” to “-” at $x = \sqrt{3}$) and $f(\sqrt{3}) = -\frac{3\sqrt{3}}{2}$ is the relative maximum

- relative minimum occurs at $x = -\sqrt{3}$ (the sign of $f'(x)$ changes from “-” to “+” at $x = -\sqrt{3}$) and $f(-\sqrt{3}) = \frac{3\sqrt{3}}{2}$ is the relative minimum. ■

1.7 Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).

From $f'(x) = \frac{3x^2-x^4}{(1-x^2)^2}$,

$$\begin{aligned} f''(x) &= \frac{(1-x^2)^2(6x-4x^3) - 2(1-x^2)(-2x)(3x^2-x^4)}{(1-x^2)^4} \\ &= 2x \frac{(1-x^2)(3-2x^2) + 6x^2 - 2x^4}{(1-x^2)^3} \\ &= \frac{2x(3+x^2)}{(1-x^2)^3} \end{aligned}$$

So $f''(0) = 0$ when $x = 0$. We will check if $x = 0$ is an inflection point, we have to see if the sign of $f''(x)$ is changed at $x = 0$ or not.

To determine the curvature, we have to solve the inequalities $f''(x) < 0$ and $f''(x) > 0$, which can be done by considering the zeros of the numerator $2x(3+x^2)$ and $(1-x^2)^3$ or at $x = 0, \pm 1$. That is, we consider the intervals $(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$.

Interval of x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of $f''(x) = \frac{2x(3+x^2)}{(1-x^2)^3}$	+	-	+	-
$\frac{2x(3+x^2)}{(1-x^2)^3}$	$\frac{(-)(+)}{(-)}$	$\frac{(-)(+)}{(-)}$	$\frac{(+)(+)}{(-)}$	$\frac{(+)(+)}{(-)}$

- Concave up: $f'' > 0$ when $x \in (-\infty, -1) \cup (0, 1)$.
- Concave down: $f'' < 0$ when $x \in (-1, 0) \cup (1, \infty)$.

From above, we have that the sign of $f''(x)$ changes at $x = 0$, so the point $(0, 0)$ is the inflection point. Note that $x = -1, 1$ don't give inflection points because they are not in the domain (even though the sign of $f''(x)$ changes at these points). ■

1.8 Sketch the curve of f .

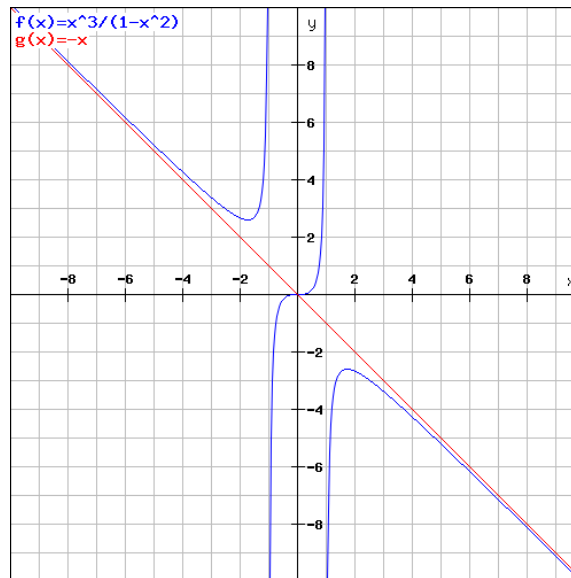


Figure 1: $f = \{(x, y) | x^3 + yx^2 - y = 0\}$

Optional Problems

1. Find the critical numbers of the given functions on $(-\infty, \infty)$.

(a) $f(x) = x^3 - 3x^2 + 3x - 1$

(b) $f(x) = \frac{x^2}{x^2+2}$

(c) $f(x) = e^{-x} + 2x$

(d) $f(x) = -x + \sin(x)$

(e) $f(x) = x^2 - 8 \ln(x)$

(a) $f(x) = x^3 - 3x^2 + 3x - 1$

Answer: $f'(x) = 3x^2 - 6x + 3$

$$f'(x) = 0 \Leftrightarrow 3(x^2 - 2x + 1) = 3(x - 1)^2 = 0 \Leftrightarrow (x - 1) = 0 \Leftrightarrow x = 1$$

Therefore, the only critical number of $f(x)$ is $x = 1$.

(b) $f(x) = \frac{x^2}{x^2+2}$

Answer:

$$f'(x) = \frac{(x^2 + 2)(2x) - x^2(2x)}{(x^2 + 2)^2} = \frac{2x^3 + 4x - x^3}{(x^2 + 2)^2} = \frac{x^3 + 4x}{(x^2 + 2)^2}$$

$$f'(x) = 0 \text{ when } x^3 + 4x = 0 \implies x(x^2 + 4) = 0 \implies x = 0.$$

Therefore, the only critical number of $f(x)$ is $x = 0$.

(c) $f(x) = e^{-x} + 2x$

Answer:

$$f'(x) = e^{-x} + 2x = -e^{-x} + 2$$

$$f'(x) = 0 \text{ when } -e^{-x} + 2 = 0 \implies e^{-x} = 2 \implies e^x = 1/2 \implies x = \ln(1/2) \text{ or } x = -\ln(2).$$

Therefore, the only critical number of $f(x)$ is $x = -\ln(2)$.

(d) $f(x) = -x + \sin(x)$

Answer:

$$f'(x) = -1 + \cos(x)$$

$$f'(x) = 0 \text{ when } -1 + \cos(x) = 0 \implies \cos(x) = 1 \implies x = 2\pi n \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Therefore, the critical numbers of $f(x)$ are $x = 2\pi n$, where $n = 0, \pm 1, \pm 2, \dots$

(e) $f(x) = x^2 - 8 \ln(x)$

Answer:

$$f'(x) = 2x - \frac{8}{x}$$

$$f'(x) = 0 \text{ when } 2x - \frac{8}{x} = 0 \implies x = \frac{4}{x} \implies x^2 = 4 \implies x = \pm 2.$$

Therefore, the critical numbers of $f(x)$ are $x = -2$ and $x = 2$.

2. Determine the intervals on which the given function f is increasing and the interval on which f is decreasing.

(a) $f(x) = x^2 + 6x - 1$

(b) $f(x) = x^4 - 4x^3 + 9$

(c) $f(x) = x^2 e^{-x}$

Answer: The function $f(x)$ is increasing when $f'(x) > 0$ and $f(x)$ is decreasing when $f'(x) < 0$.

(a) $f(x) = x^2 + 6x - 1 \implies f'(x) = 2x + 6$

$$f'(x) = 2x + 6 > 0 \implies x > -3$$

so $f(x)$ is increasing on the interval $[-3, \infty)$, and

$$f'(x) = 2x + 6 < 0 \implies x < -3$$

so $f(x)$ is decreasing on the interval $(-\infty, -3]$.

(b) $f(x) = x^4 - 4x^3 + 9 \implies f'(x) = 4x^3 - 12x^2$

$$f'(x) = 4x^3 - 12x^2 > 0 \implies 4x^2(x - 3) > 0 \implies x > 3$$

so $f(x)$ is increasing on the interval $[3, \infty)$, and

$$f'(x) = 4x^3 - 12x^2 < 0 \implies 4x^2(x - 3) < 0 \implies x < 3$$

so $f(x)$ is decreasing on the interval $(-\infty, 3]$.

(c) $f(x) = x^2 e^{-x} \implies f'(x) = -x^2 e^{-x} + 2x e^{-x} \implies f'(x) = x e^{-x} (2 - x)$,

since $e^{-x} > 0$, $f'(x) = 0 \implies x = 0$ or $x = 2$:

	$x < 0$	$0 < x < 2$	$2 < x$
Sign of $f'(x) = x e^{-x} (2 - x)$	-	+	-
$(x)(e^{-x})(2 - x)$	(-)(+)(+)	(+)(+)(+)	(+)(+)(-)

so $f(x)$ is increasing ($f'(x) = +$) on the interval $[0, 2]$,and $f(x)$ is decreasing ($f'(x) = -$) on the interval $(\infty, 0] \cup [2, \infty)$.3. Use the *First Derivative Test* to find the relative extrema of the given function.

(a) $f(x) = x^3 - 3x$

(b) $f(x) = x^3 + x - 3$

(c) $\frac{x^2+3}{x+1}$

(d) $f(x) = x^2 - 2|x|$

Answer:

(a) $f(x) = x^3 - 3x \implies f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$,

 $f'(x) = 0 \implies x = \pm 1$ are critical numbers:

	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of $f'(x) = 3(x - 1)(x + 1)$	+	-	+
$(x - 1)(x + 1)$	(-)(-)	(+)(-)	(+)(+)

Therefore, from the *First Derivative Test*,- relative maximum occurs at $x = -1$ (“+” changes to “-” at $x = -1$)and $f(-1) = (-1)^3 - 3(-1) = 2$ is the relative maximum- relative minimum occurs at $x = 1$ (“-” changes to “+” at $x = 1$) and $f(1) = 1^3 - 3(1) = -2$ is the relative minimum.

(b) $f(x) = x^3 + x - 3 \implies f'(x) = 3x^2 + 1 \neq 0, \forall x \in (-\infty, \infty)$.

$f'(x)$ is well-defined for any x and $f'(x) \neq 0$ imply that there is *no critical number* for $f(x)$. Since a relative extremum can only occur at a critical number and there is no critical number, therefore, we conclude that **there is no relative extrema** for this function.

- (c) $f(x) = \frac{x^2+3}{x+1}$ Note that the domain of this function is all real number except for $x = -1$:
 $D = \mathbb{R}/\{-1\}$
 $\implies f'(x) = \frac{(x+1)(2x)-(x^2+3)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$,
 $f'(x) = 0 \implies (x+3)(x-1) = 0 \implies x = -3, 1$ are the critical numbers.

	$x < -3$	$-3 < x < 1 (x \neq -1)$	$1 < x$
Sign of $f'(x)$	+	-	+
$(x+3)(x-1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,

- relative maximum occurs at $x = -3$ (“+” changes to “-” at $x = -3$)

and $f(-3) = \frac{(-3)^2+3}{-3+1} = -6$ is the relative maximum

- relative minimum occurs at $x = 1$ (“-” changes to “+” at $x = 1$) and

$f(1) = \frac{1^2+3}{1+1} = 2$ is the relative minimum.

Notice that it is possible to have a “relative” minimum *larger* than a “relative” maximum.

- (d) Note that we can write $f(x) = x^2 - 2|x|$ as:

- For $x < 0$, $f(x) = x^2 + 2x$

- For $x \geq 0$, $f(x) = x^2 - 2x$.

- (i) The critical numbers:

- For $x < 0$, $f'(x) = 2x + 2$

- For $x > 0$, $f'(x) = 2x - 2$,

and $f'(x) = 0$ when $2x + 2 \Rightarrow x = -1$ or $2x - 2 = 0 \Rightarrow x = 1$. Note that $f'(x)$ does not exist when $x = 0$. Hence, the critical numbers are $x = -1, 1, 0$.

- (ii) Relative extrema: Relative extrema can occur only at the critical points: $x = -1, 1, 0$.

So we consider the subintervals:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
$f'(x)$	$f'(x) = 2x + 2$	$f'(x) = 2x + 2$	$f'(x) = 2x - 2$	$f'(x) = 2x - 2$
Sign of $f'(x)$	-	+	-	+

Therefore, from the *First Derivative Test*,

- relative maximum occurs at $x = 0$ (“+” changes to “-” at $x = 0$)

and $f(0) = 0$ is the relative maximum

- relative minima occurs at $x = -1, 1$ (“-” changes to “+” at $x = -1, 1$) and

$f(-1) = 1^2 - 2|-1| = -1$ and $f(1) = 1^2 - 2|1| = -1$ are the relative minima.

That is, -1 is the relative minimum, which occurs at $x = \pm 1$.

4. For each given function,

(i) use the Second Derivative Test, when applicable, to find the relative extrema;

(ii) find intercepts and points of inflection, when possible;

(iii) find the intervals on which it is increasing and the interval on which it is decreasing;

(iv) find the intervals on which it is concave up and the intervals on which it is concave down;

and (v) sketch the graph.

- (a) $f(x) = x^3 + 3x^2 + 3x + 1$
 (b) $f(x) = 6x^5 - 10x^3$
 (c) $f(x) = \frac{x}{x^2+2}$
 (d) $f(x) = 2x - x \ln(x)$

Answer:

Note that there are 2 steps

(1) Find critical numbers c_j

(2) Check sign of $f''(c_j)$:

- If $f''(c_j) < 0$, $f(c_j)$ is a relative maximum.

- If $f''(c_j) > 0$, $f(c_j)$ is a relative minimum.

(If $f''(c_j) = 0$, no conclusion – use the First Derivative Test.)

- (a) $f(x) = x^3 + 3x^2 + 3x + 1$

$$f'(x) = 3x^2 + 6x + 3, \quad f''(x) = 6x + 6.$$

(i) Critical numbers:

$$f'(x) = 3x^2 + 6x + 3 = 0 \quad \Rightarrow \quad 3(x+1)^2 = 0 \quad \Rightarrow \quad x = -1 \text{ is the only critical number.}$$

From the Second Derivative Test,

$$f''(-1) = 6(-1) + 6 = 0$$

so we cannot conclude anything. The First derivative will be used instead,

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f'(x) = 3(x+1)^2$	+	+

Therefore, from the table above, the *First Derivative Test* implies that there is **no relative extremum** since the sign of $f'(x)$ is always positive.

(ii) Find intercepts and points of inflection:

- The y-intercept($x=0$) is at $y = f(0) = 0^3 + 3 \cdot 0^2 + 3(0) + 1 = 1 \Rightarrow y = 1$.

- The x-intercept($y=0$) is at $0 = f(x) = (x+1)^3 \Rightarrow x = -1$.

- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 6x + 6 = 0 \quad \Rightarrow \quad x = -1.$$

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f''(x) = 6(x+1)$	-	+

Since the sign of $f''(x)$ changes at the $x = -1$, it gives an inflection point. When $x = -1$, $y = f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = 0$. Hence, the inflection point is $(-1, 0)$.

(iii) Since $f'(x) = 3(x+1)^2 > 0$ for all $x \neq -1$, and $f'(x) = 0$ at $x = -1$, $f(x)$ is increasing on $(-\infty, \infty)$.

(iv) Concavity:

$$f''(x) = 6x + 6 > 0 \Rightarrow x > -1 \text{ and } f''(x) = 6x + 6 < 0 \Rightarrow x < -1.$$

That is, $f(x)$ is concave up on $(-1, \infty)$ and $f(x)$ is concave down on $(-\infty, -1)$.

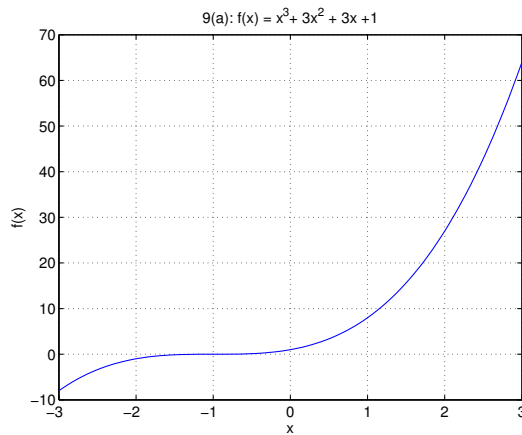


Figure 2: (a): $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = 6x^5 - 10x^3$

(i) Critical numbers:

$$f'(x) = 30x^4 - 30x^2 = 30x^2(x^2 - 1) = 30x^2(x - 1)(x + 1) = 0 \Rightarrow$$

 $x = 0, -1, 1$ are the only critical numbers. From

$$f''(x) = 120x^3 - 60x = 60x(2x^2 - 1)$$

and from the Second Derivative Test,

$f''(0) = 0 \Rightarrow$, no conclusion

$f''(-1) = -60 < 0 \Rightarrow$, $f(-1) = -6 + 10 = 4$ is a relative maximum

$f''(1) = 60 > 0 \Rightarrow$, $f(1) = 6 - 10 = -4$ is a relative minimum.

The First Derivative Test will be used to identify the critical number $x = 0$ as shown in the table. Since the sign of $f'(x)$ does not change at $x = 0$, $x = 0$ does not give an extremum.

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1)$	$x \in (1, \infty)$
Sign of $f'(x) = 30x^2(x - 1)(x + 1)$ $x^2(x - 1)(x + 1)$	+	-	-	+
	(+)(-)(-)	(+)(-)(+)	(+)(-)(+)	(+)(+)(+)

(ii) Find intercepts and points of inflection:

- The y-intercept ($x=0$) is at $y = f(0) = 6(0)^5 - 10(0)^3 = 0 \Rightarrow y = 0$.

- The x-intercept ($y=0$) occurs when $0 = f(x) = x^3(6x^2 - 10)$

\Rightarrow at $x = 0, x = -\sqrt{5/3}, x = \sqrt{5/3}$.

- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 60x(2x^2 - 1) = 0 \Rightarrow x = 0, -\sqrt{1/2}, \sqrt{1/2}.$$

	$x \in (-\infty, -\sqrt{1/2})$	$x \in (-\sqrt{1/2}, 0)$	$x \in (0, \sqrt{1/2})$	$x \in (\sqrt{1/2}, \infty)$
Sign of $f''(x) = 60x(2x^2 - 1)$ $(x)(x - \sqrt{1/2})(x + \sqrt{1/2})$	-	+	-	+
	(-)(-)(+)	(-)(-)(+)	(+)(-)(+)	(+)(+)(+)

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{1/2}, \sqrt{1/2}$, they give inflection points.

Since $f(0) = 0, f(-\sqrt{1/2}) = 7\sqrt{2}/4, f(\sqrt{1/2}) = -7\sqrt{2}/4,$

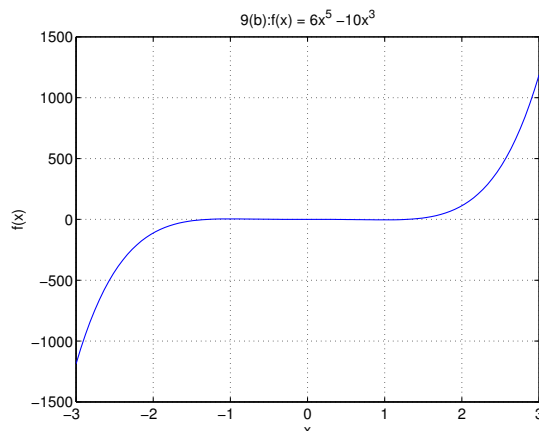
the inflection points are $(0, 0), (-\sqrt{1/2}, 7\sqrt{2}/4), (\sqrt{1/2}, -7\sqrt{2}/4)$.

(iii) From the table in (i), we see that

 $f(x)$ is increasing ($f'(x) > 0$) on $(-\infty, -1) \cup (1, \infty)$ and $f(x)$ is decreasing ($f'(x) < 0$) on $(-1, 1)$.

(iv) From the table in (ii), we see that

 $f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{1/2}, 0) \cup (\sqrt{1/2}, \infty)$ and $f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{1/2}) \cup (0, \sqrt{1/2})$.

Figure 3: (b): $f(x) = 6x^5 - 10x^3$

(c) $f(x) = \frac{x}{x^2+2}$

(i) Critical numbers:

$$f'(x) = \frac{(x^2 + 2) - 2x^2}{(x^2 + 2)^2} = \frac{2 - x^2}{(x^2 + 2)^2} = 0 \Rightarrow 2 - x^2 = 0$$

$x = -\sqrt{2}, \sqrt{2}$ are the only critical numbers. From

$$f''(x) = \frac{2x(x^2 + 2)(x^2 - 6)}{(x^2 + 2)^4} = \frac{2x(x^2 - 6)}{(x^2 + 2)^3}$$

and from the Second Derivative Test,

$$f''(-\sqrt{2}) = \frac{\sqrt{2}}{8} > 0 \Rightarrow f(-\sqrt{2}) = \frac{-\sqrt{2}}{(-\sqrt{2})^2+2} = -\frac{\sqrt{2}}{4} \text{ is a relative minimum}$$

$$f''(\sqrt{2}) = -\frac{\sqrt{2}}{8} < 0 \Rightarrow f(\sqrt{2}) = \frac{\sqrt{2}}{(\sqrt{2})^2+2} = \frac{\sqrt{2}}{4} \text{ is a relative maximum.}$$

(ii) Find intercepts and points of inflection:

- The y-intercept ($x=0$) is at $y = f(0) = 0 \Rightarrow y = 0$.

- The x-intercept ($y=0$) occurs when $0 = f(x) \Rightarrow$ at $x = 0$.

- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = \frac{2x(x^2 - 6)}{(x^2 + 2)^3} = 0 \Rightarrow x = 0, -\sqrt{6}, \sqrt{6}.$$

Since the denominator $(x^2 + 2)^3 > 0$, the sign of $f''(x)$ depends on the numerator $2x(x^2 - 6) = 2x(x - \sqrt{6})(x + \sqrt{6})$ as shown in the table below.

	$x \in (-\infty, -\sqrt{6})$	$x \in (-\sqrt{6}, 0)$	$x \in (0, \sqrt{6})$	$x \in (\sqrt{6}, \infty)$
Sign of $f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3}$	-	+	-	+
$(2x)(x - \sqrt{6})(x + \sqrt{6})$	$(-)(-)(+)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{6}, \sqrt{6}$, they give inflection points. Since $f(0) = 0$, $f(-\sqrt{6}) = \frac{-\sqrt{6}}{(-\sqrt{6})^2+2} = -\sqrt{6}/8$, $f(\sqrt{6}) = \frac{\sqrt{6}}{(\sqrt{6})^2+2} = \sqrt{6}/8$, the inflection points are $(0, 0)$, $(-\sqrt{6}, -\sqrt{6}/8)$, $(\sqrt{6}, \sqrt{6}/8)$.

(iii) From

$$f'(x) = \frac{2 - x^2}{(x^2 + 2)^2} = \frac{(\sqrt{2} - x)(\sqrt{2} + x)}{(x^2 + 2)^2},$$

the denominator is always positive, so the sign of $f'(x)$ depends on the numerator: $(\sqrt{2} - x)(\sqrt{2} + x)$.

	$x \in (-\infty, -\sqrt{2})$	$x \in (-\sqrt{2}, \sqrt{2})$	$x \in (\sqrt{2}, \infty)$
Sign of $f'(x) = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2}$	-	+	-
$(\sqrt{2} - x)(\sqrt{2} + x)$	(+)(-)	(+)(+)	(-)(+)

Hence, $f(x)$ is increasing ($f'(x) > 0$) on $(-\sqrt{2}, \sqrt{2})$ and $f(x)$ is decreasing ($f'(x) < 0$) on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

(iv) From the table in (ii), we see that $f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$ and $f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$.

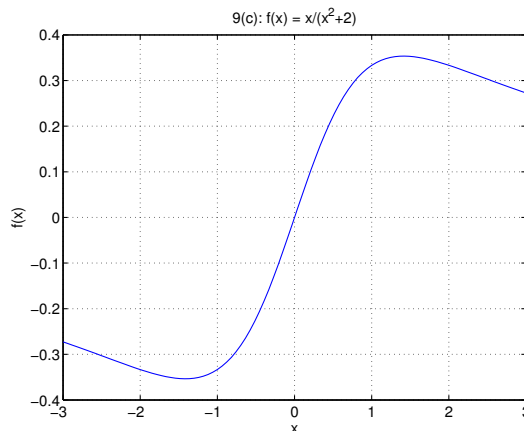


Figure 4: (c): $f(x) = \frac{x}{x^2+2}$

(d) $f(x) = 2x - x \ln(x)$

First note that the domain of f is

$$\mathcal{D} = (0, \infty),$$

since $\ln(x)$ is only well-defined for $x > 0$.

(i) Critical numbers: $f'(x) = 2 - [x^{\frac{1}{x}} + \ln(x)] = 1 - \ln(x)$.

$$f'(x) = 1 - \ln(x) = 0 \Rightarrow \ln(x) = 1 \Rightarrow x = e^1.$$

$x = e$ is the only critical number. From

$$f''(x) = -\frac{1}{x}$$

and from the Second Derivative Test,

$$f''(e) = -\frac{1}{e} < 0 \implies, f(e) = 2e - e \ln(e) = 2e - e = e \text{ is a relative maximum.}$$

(ii) Find intercepts and points of inflection:

- There is no y-intercept, since $x = 0$ is not in the domain of the function $f(x) = 2x - x \ln(x)$ ($\ln(x)$ is not well-defined at $x = 0$).

- The x-intercept ($y=0$) occurs when $0 = f(x) \Rightarrow 2x - x \ln(x) = x(2 - \ln(x)) = 0 \Rightarrow x = 0$ or $\ln(x) = 2 \Rightarrow$ at $x = 0, x = e^2$.

- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$: notice that

$$f''(x) = -\frac{1}{x} \neq 0 \implies \text{There is no inflection point.}$$

(iii) From $f'(x) = 1 - \ln(x)$,

- $f'(x) > 0$ when $1 - \ln(x) > 0 \Leftrightarrow \ln(x) < 1 \Leftrightarrow x < e \Rightarrow f(x)$ is **increasing** on $(-\infty, e]$.
- $f'(x) < 0$ when $1 - \ln(x) < 0 \Leftrightarrow \ln(x) > 1 \Leftrightarrow x > e \Rightarrow f(x)$ is **decreasing** on $[e, \infty)$.

(iv) From $f''(x) = -\frac{1}{x}$,

$f(x)$ is concave down ($f''(x) < 0 \Leftrightarrow x > 0$) on $(0, \infty)$. Note that $f(x)$ is not concave up, since $f''(x) > 0 \Leftrightarrow x < 0$, but $x < 0$ is not in the domain of $f(x)$: $\mathcal{D} = (0, \infty)$.

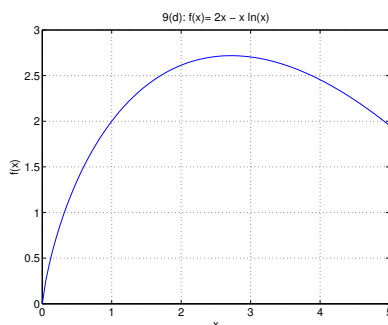


Figure 5: (d): $f(x) = 2x - x \ln(x)$

5. Let $X, Y_1, Y_2 \subseteq \mathbb{R}$. Consider the relations f_1, f_2 defined by

$$f_1 = \left\{ (x, y) \in X \times Y_1 \mid y = \sqrt{\frac{x+1}{2x-3}} \right\}, \quad f_2 = \left\{ (x, y) \in X \times Y_2 \mid (x, -y) \in f_1 \right\}.$$

Sketch the curve of the relation $f = f_1 \cup f_2$.

Answer:

(Details to be added...symmetry/critical numbers/increasing& decreasing/ concavity...)

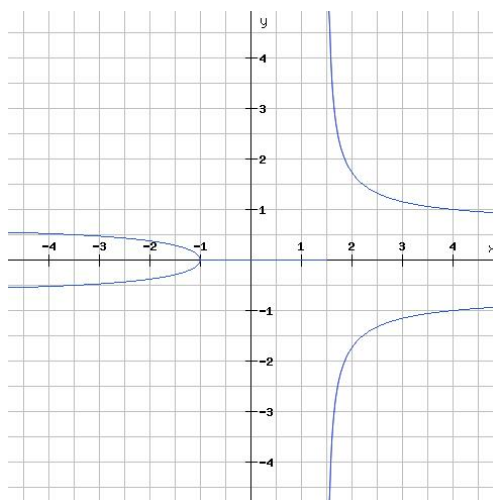


Figure 6: $f = f_1 \cup f_2 = \{(x, y) \mid y^2 = \frac{x+1}{2x-3}\}$

6. Find the vertical and horizontal asymptotes (if any) of $f(x) = \frac{x^2-1}{2x}$ and show that $y = \frac{x}{2}$ is a slant asymptote of f .

Answer:(Details to be added...)

- Vertical asymptote: $x = 0$
- Horizontal asymptote: None
- Slant asymptote $y = \frac{x}{2}$. We have to show that $\lim_{x \rightarrow \infty} [f(x) - \frac{x}{2}] = 0$.

$$\lim_{x \rightarrow \infty} [f(x) - \frac{x}{2}] = \lim_{x \rightarrow \infty} \left(\frac{x^2-1}{2x} - \frac{x}{2} \right) = \lim_{x \rightarrow \infty} \left(\frac{x^2-1-x^2}{2x} \right) = \lim_{x \rightarrow \infty} \frac{-1}{2x} = 0$$