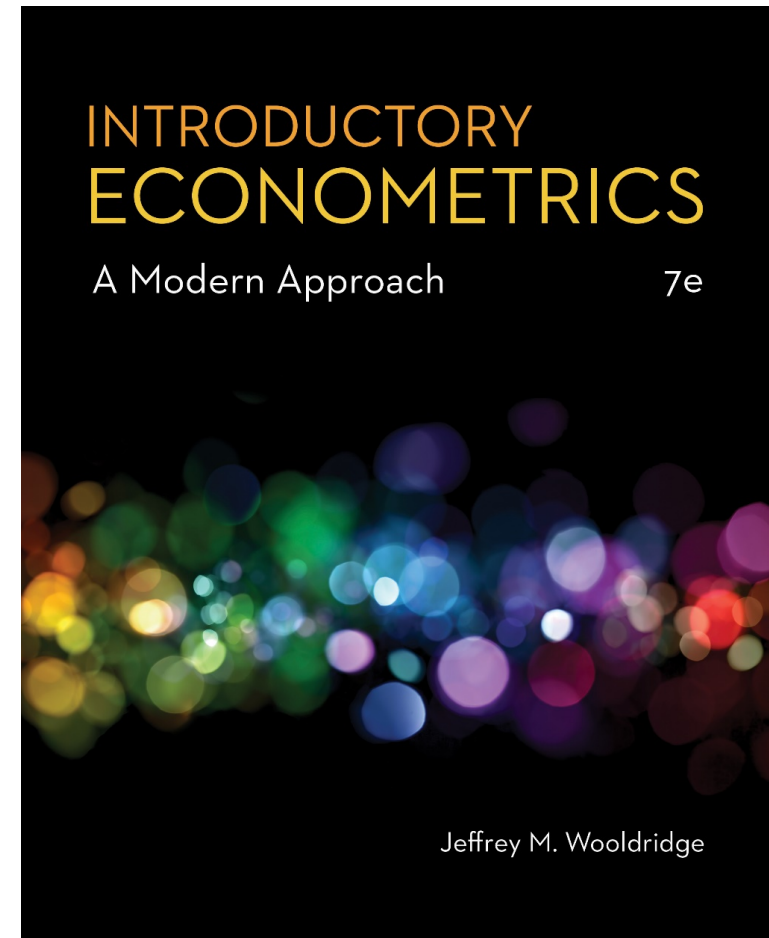


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Chapter 8

Heteroskedasticity



Heteroskedasticity (1 of 18)

- **Consequences of heteroskedasticity for OLS**

- OLS still unbiased and consistent under heteroskedasticity!
- Also, interpretation of R-squared is not changed

$$R^2 \approx 1 - \frac{\sigma_u^2}{\sigma_y^2}$$

← Unconditional error variance is unaffected by heteroskedasticity (which refers to the conditional error variance)

- Heteroskedasticity invalidates variance formulas for OLS estimators
- The usual F tests and t tests are not valid under heteroskedasticity
- Under heteroskedasticity, OLS is no longer the best linear unbiased estimator (BLUE); there may be more efficient linear estimators

Heteroskedasticity (2 of 18)

• Heteroskedasticity-robust inference after OLS estimation

- Formulas for OLS standard errors and related statistics have been developed that are robust to heteroskedasticity of unknown form.
- All formulas are only valid in large samples.
- Formula for heteroskedasticity-robust OLS standard error.

$$\widehat{Var}(\hat{\beta}_j) = \frac{\sum_{i=1}^n \hat{r}_{ij}^2 \hat{u}_i^2}{SSR_j^2}$$

Also called White/Huber/Eicker standard errors. They involve the squared residuals from the regression and from a regression of x_j on all other explanatory variables.

- Using these formulas, the usual t test is valid asymptotically.
- The usual F statistic does not work under heteroskedasticity, but heteroskedasticity robust versions are available in most software.

Heteroskedasticity (3 of 18)

• Example: Hourly wage equation

$$\widehat{\log(wage)} = - .128 + .0904 \text{ educ} + .0410 \text{ exper} - .0007 \text{ exper}^2$$

(.105)	(.0075)	(.0052)	(.0001)
[.107]	[.0078]	[.0050]	[.0001]

Heteroskedasticity robust standard errors may be larger or smaller than their nonrobust counterparts. The differences are often small in practice.

$$H_0 : \beta_{\text{exper}} = \beta_{\text{exper}^2} = 0$$

$$F = 17.95$$

F statistics are also often not too different.

$$F_{\text{robust}} = 17.99$$

If there is strong heteroskedasticity, differences may be larger. To be on the safe side, it is advisable to always compute robust standard errors.

Heteroskedasticity (4 of 18)

- **Testing for heteroskedasticity**

- It may still be interesting whether there is heteroskedasticity because then OLS may not be the most efficient linear estimator anymore.

- Breusch-Pagan test for heteroskedasticity

$$H_0 : Var(u|x_1, x_2, \dots, x_k) = Var(u|\mathbf{x}) = \sigma^2$$

$$Var(u|\mathbf{x}) = E(u^2|\mathbf{x}) - [E(u|\mathbf{x})]^2 = E(u^2|\mathbf{x})$$

Under MLR.4

$$\Rightarrow E(u^2|x_1, \dots, x_k) = E(u^2) = \sigma^2$$

The mean of u^2 must not vary with $x_1, x_2, \dots, x_k$

Heteroskedasticity (5 of 18)

• Breusch-Pagan test for heteroskedasticity (cont.)

$$\hat{u}^2 = \delta_0 + \delta_1 x_1 + \cdots + \delta_k x_k + error$$

$$H_0 : \delta_1 = \delta_2 = \cdots = \delta_k = 0 \leftarrow \text{Regress squared residuals on all explanatory variables and test whether this regression has explanatory power.}$$

$$F = \frac{R_{\hat{u}^2}^2 / k}{1 - R_{\hat{u}^2}^2 / (n - k - 1)} \leftarrow \text{A large test statistic (= a high R-squared) is evidence against the null hypothesis.}$$

$$LM = n \cdot R_{\hat{u}^2}^2 \sim \chi_k^2 \leftarrow \text{Alternative test statistic (= Lagrange multiplier statistic, LM). Again, high values of the test statistic (= high R-squared) lead to rejection of the null hypothesis that the expected value of } u^2 \text{ is unrelated to the explanatory variables.}$$

Heteroskedasticity (6 of 18)

• Example: Heteroskedasticity in housing price equations

$$\widehat{price} = -21.77 + .0021 \text{ lotsize} + .123 \text{ sqrft} + 13.85 \text{ bdrms}$$

(29.48) (.0006) (.013) (9.01)

$$\Rightarrow R_{\hat{u}^2}^2 = .1601, p\text{-value}_F = .002, p\text{-value}_{LM} = .0028 \leftarrow \text{Heteroskedasticity}$$

$$\widehat{\log(price)} = -1.30 + .168 \log(lotsize) + .700 \log(sqrft) + .037 \text{ bdrms}$$

(.65) (.038) (.093) (.028)

$$\Rightarrow R_{\hat{u}^2}^2 = .0480, p\text{-value}_F = .245, p\text{-value}_{LM} = .2390$$

↑
In the logarithmic specification, homoskedasticity cannot be rejected

Heteroskedasticity (7 of 18)

• The White test for heteroskedasticity

$$\hat{u}^2 = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + \delta_4 x_1^2 + \delta_5 x_2^2 + \delta_6 x_3^2 + \delta_7 x_1 x_2 + \delta_8 x_1 x_3 + \delta_9 x_2 x_3 + error$$

← Regress squared residuals on all explanatory variables, their squares, and in-teractions (here: example for k=3)

$$H_0 : \delta_1 = \delta_2 = \dots = \delta_9 = 0$$

← The White test detects more general deviations from heteroskedasticity than the Breusch-Pagan test

$$LM = n \cdot R_{\hat{u}^2}^2 \sim \chi_9^2$$

←

- Disadvantage of this form of the White test
 - Including all squares and interactions leads to a large number of estimated parameters (e.g. k=6 leads to 27 parameters to be estimated).

Heteroskedasticity (8 of 18)

- **Alternative form of the White test**

$$\hat{u}^2 = \delta_0 + \delta_1 \hat{y} + \delta_2 \hat{y}^2 + error$$



This regression indirectly tests the dependence of the squared residuals on the explanatory variables, their squares, and interactions, because the predicted value of y and its square implicitly contain all of these terms.

- **Example: Heteroskedasticity in (log) housing price equations**

$$H_0 : \delta_1 = \delta_2 = 0, \quad LM = n \cdot R_{\hat{u}^2}^2 \sim \chi_2^2$$

$$R_{\hat{u}^2}^2 = .0392, \quad LM = 88(.0392) \approx 3.45, \quad p\text{-value}_{LM} = .178$$

Heteroskedasticity (9 of 18)

- **Weighted least squares estimation**
- Heteroskedasticity is known up to a multiplicative constant

$$Var(u_i|x_i) = \sigma^2 h(x_i), \quad h(x_i) = h_i > 0 \quad \leftarrow \text{The functional form of the heteroskedasticity is known}$$

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + u_i$$

$$\Rightarrow \left[\frac{y_i}{\sqrt{h_i}} \right] = \beta_0 \left[\frac{1}{\sqrt{h_i}} \right] + \beta_1 \left[\frac{x_{i1}}{\sqrt{h_i}} \right] + \cdots + \beta_k \left[\frac{x_{ik}}{\sqrt{h_i}} \right] + \left[\frac{u_i}{\sqrt{h_i}} \right]$$

$$\Leftrightarrow y_i^* = \beta_0 x_{i0}^* + \beta_1 x_{i1}^* + \cdots + \beta_k x_{ik}^* + u_i^* \quad \leftarrow \text{Transformed model}$$

Heteroskedasticity (10 of 18)

- **Example: Savings and income**

$$sav_i = \beta_0 + \beta_1 inc_i + u_i, \quad Var(u_i | inc_i) = \sigma^2 inc_i$$

$$\left[\frac{sav_i}{\sqrt{inc_i}} \right] = \beta_0 \left[\frac{1}{\sqrt{inc_i}} \right] + \beta_1 \left[\frac{inc_i}{\sqrt{inc_i}} \right] + u_i^* \quad \leftarrow \text{Note that this regression model has no intercept}$$

- The transformed model is homoskedastic

$$E(u_i^{*2} | \mathbf{x}_i) = E \left[\left(\frac{u_i}{\sqrt{h_i}} \right)^2 | \mathbf{x}_i \right] = \frac{E(u_i^2 | \mathbf{x})}{h_i} = \frac{\sigma^2 h_i}{h_i} = \sigma^2$$

- If the other Gauss-Markov assumptions hold as well, OLS applied to the transformed model is the best linear unbiased estimator.

Heteroskedasticity (11 of 18)

- **OLS in the transformed model is weighted least squares (WLS)**

$$\min \sum_{i=1}^n \left(\left[\frac{y_i}{\sqrt{h_i}} \right] - b_0 \left[\frac{1}{\sqrt{h_i}} \right] - b_1 \left[\frac{x_{i1}}{\sqrt{h_i}} \right] - \dots - b_k \left[\frac{x_{ik}}{\sqrt{h_i}} \right] \right)^2$$

$$\Leftrightarrow \min \sum_{i=1}^n (y_i - b_0 - b_1 x_{i1} - \dots - b_k x_{ik})^2 / h_i$$

← Observations with a large variance get a smaller weight in the optimization problem

- Why is WLS more efficient than OLS in the original model?
 - Observations with a large variance are less informative than observations with small variance and therefore should get less weight.
- WLS is a special case of generalized least squares (GLS)

Heteroskedasticity (12 of 18)

• Example: Financial wealth equation

Net financial wealth

TABLE 8.1 Dependent Variable: *nettfa*

Independent Variables	(1) OLS	(2) WLS	(3) OLS	(4) WLS
<i>inc</i>	.821 (.104)	.787 (.063)	.771 (.100)	.740 (.064)
$(age - 25)^2$	—	—	.0251 (.0043)	.0175 (.0019)
<i>male</i>	—	—	2.48 (2.06)	1.84 (1.56)
<i>e401k</i>	—	—	6.89 (2.29)	5.19 (1.70)
<i>intercept</i>	-10.57 (2.53)	-9.58 (1.65)	-20.98 (3.50)	-16.70 (1.96)
Observations	2,017	2,017	2,017	2,017
<i>R</i> -squared	.0827	.0709	.1279	.1115

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Assumed form of heteroskedasticity

WLS estimates have considerably smaller standard errors (which is line with the expectation that they are more efficient).

Participation in 401K pension plan

Heteroskedasticity (13 of 18)

- Important special case of heteroskedasticity
 - If the observations are reported as averages at the city/county/state/-country/firm level, they should be weighted by the size of the unit

Average contribution to pension plan in firm i Average earnings and age in firm i Percentage firm contributes to plan Heteroskedastic error term

$$\overline{contrib}_i = \beta_0 + \beta_1 \overline{earns}_i + \beta_2 \overline{age}_i + \beta_3 \overline{mrate}_i + \overline{u}_i$$

$$\Rightarrow Var(\overline{u}_i) = Var\left(\frac{1}{m_i} \sum_{e=1}^{m_i} u_{i,e}\right) = \sigma^2 / m_i$$

Error variance if errors are homoskedastic at the individual-level

- If errors are homoskedastic at the individual-level, WLS with weights equal to firm size m_i should be used. If the assumption of homoskedasticity at the individual-level is not exactly right, one can calculate robust standard errors after WLS (i.e. for the transformed model).

Heteroskedasticity (14 of 18)

• Unknown heteroskedasticity function (feasible GLS)

$$Var(u|x) = \sigma^2 \exp(\delta_0 + \delta_1 x_1 + \dots + \delta_k x_k) = \sigma^2 h(x)$$

← Assumed general form of heteroskedasticity; exp-function is used to ensure positivity

$$u^2 = \sigma^2 \exp(\delta_0 + \delta_1 x_1 + \dots + \delta_k x_k) \cdot v$$

← Multiplicative error (assumption: independent of the explanatory variables)

$$\Rightarrow \log(u^2) = \alpha_0 + \delta_1 x_1 + \dots + \delta_k x_k + e$$

$$\log(\hat{u}^2) = \hat{\alpha}_0 + \hat{\delta}_1 x_1 + \dots + \hat{\delta}_k x_k + error$$

$$\Rightarrow \hat{h}_i = \exp(\hat{\alpha}_0 + \hat{\delta}_1 x_1 + \dots + \hat{\delta}_k x_k)$$

← Use inverse values of the estimated heteroskedasticity function as weights in WLS

Heteroskedasticity (15 of 18)

- **Example: Demand for cigarettes**
- Estimation by OLS

Cigarettes smoked per day Logged income and cigarette price

$$\widehat{cigs} = - \frac{3.64}{(24.08)} + \frac{.880}{(.728)} \log(income) - \frac{.751}{(5.773)} \log(cigpric)$$

$$- \frac{.501}{(.167)} educ - \frac{.771}{(.160)} age - \frac{.0090}{(.0017)} age^2 - \frac{2.83}{(1.11)} restaurn$$

Smoking restrictions in restaurants

$$n = 807, R^2 = .0526, p\text{-value}_{Breusch-Pagan} = .000 \leftarrow \text{Reject homoskedasticity}$$

Heteroskedasticity (16 of 18)

• Estimation by FGLS

Now statistically significant


$$\widehat{cigs} = - \frac{5.64}{(17.80)} + \frac{1.30}{(.44)} \log(income) - \frac{2.94}{(4.46)} \log(cigpric)$$

$$- \frac{.463}{(.120)} educ + \frac{.482}{(.097)} age - \frac{.0056}{(.0009)} age^2 - \frac{3.46}{(.80)} restaurn$$

$$n = 807, R^2 = .1134$$

• Discussion

- The income elasticity is now statistically significant; other coefficients are also more precisely estimated (without changing qualitative results).

Heteroskedasticity (17 of 18)

- **What if the assumed heteroskedasticity function is wrong?**
 - If the heteroskedasticity function is misspecified, WLS is still consistent under MLR.1 – MLR.4, but robust standard errors should be computed.
 - WLS is consistent under MLR.4 but not necessarily under MLR.4'
- If OLS and WLS produce very different estimates, this typically indicates that some other assumptions (e.g. MLR.4) are wrong.
- If there is strong heteroskedasticity, it is still often better to use a wrong form of heteroskedasticity in order to increase efficiency.

Heteroskedasticity (18 of 18)

• WLS in the linear probability model

$$P(y = 1|\mathbf{x}) = p(\mathbf{x}) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$$

$$\Rightarrow \text{Var}(y|\mathbf{x}) = p(\mathbf{x}) [1 - p(\mathbf{x})] \leftarrow \text{In the LPM, the exact form of heteroskedasticity is known}$$

$$\Rightarrow \hat{h}_i = \hat{y}_i(1 - \hat{y}_i) \leftarrow \text{Use inverse values as weights in WLS}$$

• Discussion

- Infeasible if LPM predictions are below zero or greater than one.
- If such cases are rare, they may be adjusted to values such as .01/.99.
- Otherwise, it is probably better to use OLS with robust standard errors.