Maximum Likelihood Estimation

Motivation

Disadvantages of Least Squares Methods

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'\varepsilon$$

If ε is large, it will have impacts on estimated coefficients.

To solve this problem,

- Transform the data.
- Apply another distribution – nonnormal.

Maximum Likelihood Estimation

Maximum Likelihood Estimation

Idea of Maximum Likelihood

- MLE stands for Maximum Likelihood Estimation
- Thus, the method is to try to maximize the likelihood function of the model.
- What is the likelihood function of the model?
- Distinctions between Probability Density function *(pdf)* and Likelihood function

Maximum Likelihood Estimation

Maximum Likelihood Estimation

Probability density function describes probability density for (y, X) treating θ as given

Likelihood function describes the situation when treating X and y as given and treating θ as variables.

$$L(\theta, y, X) \equiv p(y, X, \theta)$$

Example

$X_1 = 4$ and $X_2 = 6$

Estimate μ

Assume: Normal distribution and $\sigma = 1$

Pdf. $$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left[-\frac{(X-\mu)^2}{2\sigma^2}\right]} = \frac{1}{\sqrt{2\pi}} e^{\left[-\frac{(X-\mu)^2}{2}\right]}$$

Likelihood function

$$L = \left(\frac{1}{\sqrt{2\pi}} e^{\left[-\frac{(4-\mu)^2}{2}\right]}\right)\left(\frac{1}{\sqrt{2\pi}} e^{\left[-\frac{(6-\mu)^2}{2}\right]}\right)$$

Example

μ	p(4|μ)	p(6|μ)	L	log L
3.5	0.3520	0.0175	0.0062	-5.0883
4.0	0.3989	0.0540	0.0215	-3.8383
4.5	0.3520	0.1295	0.0456	-3.0883
4.6	0.3332	0.1497	0.0499	-2.9983
4.7	0.3122	0.1713	0.0535	-2.9283
4.8	0.2896	0.1941	0.0562	-2.8783
4.9	0.2660	0.2178	0.0579	-2.8483
5.0	0.2419	0.2419	0.0585	-2.8383
5.1	0.2178	0.2660	0.0579	-2.8483
5.2	0.1941	0.2896	0.0562	-2.8783
5.3	0.1713	0.3122	0.0535	-2.9283
5.4	0.1497	0.3332	0.0499	-2.9983
5.5	0.1295	0.3520	0.0456	-3.0883
6.0	0.0540	0.3989	0.0215	-3.8383
6.5	0.0175	0.3520	0.0062	-5.0883

Example

$$L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_i\left(Y_i - \beta_1 - \beta_2 X_i\right)^2$$

MLE

β_2 =	0.5091	β_1 =	24.4545	log Likelihood	-16.1162294
σ^2=	42.1591			Likelihood	0.0000001002

OLS Result

β_2 =	0.50909	β_1 =	24.4545	r^2 =	0.96206156
var(β_2) =	0.00128	var(β_1) =	41.1371	σ^2=	42.15909091
se(β_2) =	0.03574	se(β_1) =	6.41382	se =	6.493003227
				log Likelihood	-16.1162294

Maximum Likelihood Estimation

Maximum Likelihood Estimation

Consider all cases: i = 1, 2, 3,…, n and assume independent for all x_i

$$L(y, X, \theta) = f(y_i, X_i | \theta) = \prod_{i=1}^{n} f(y_i, X_i, \theta)$$

Loglikelihood function can be defined as

$$l(\theta) = \log(L(\theta))$$

For example:

$$l(\beta, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)$$

Maximum Likelihood Computation

Maximum Likelihood Estimation

Numerical Aspects of Optimization

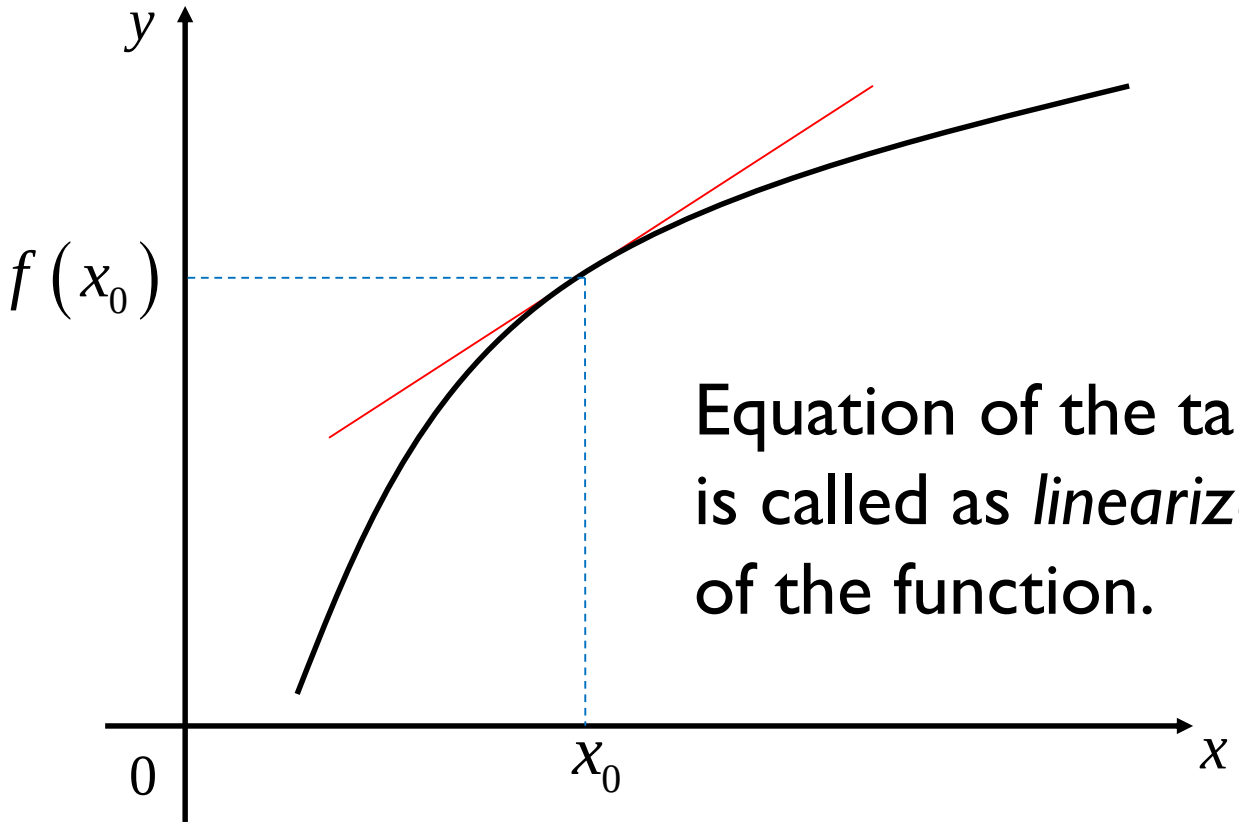
Non-linear Optimization:

1. Linearize non-linear function using Taylor Linear Approximation.
2. Min. or Max. the Linear-Approximation of the non-linear objective function → find the FOC.

Maximum Likelihood Computation

Linearize at Point x_0

For any function $f(x)$, the tangent is a close approximation of the function for some small distance from the tangent point.



Equation of the tangent is called as *linearization* of the function.

Maximum Likelihood Computation

Linearize at Point x_0

Start with the point/slope equation:

$$y - y_0 = m(x - x_0) \qquad y_0 = f(x_0) \qquad m = f'(x_0)$$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$ linearization of f at x_0

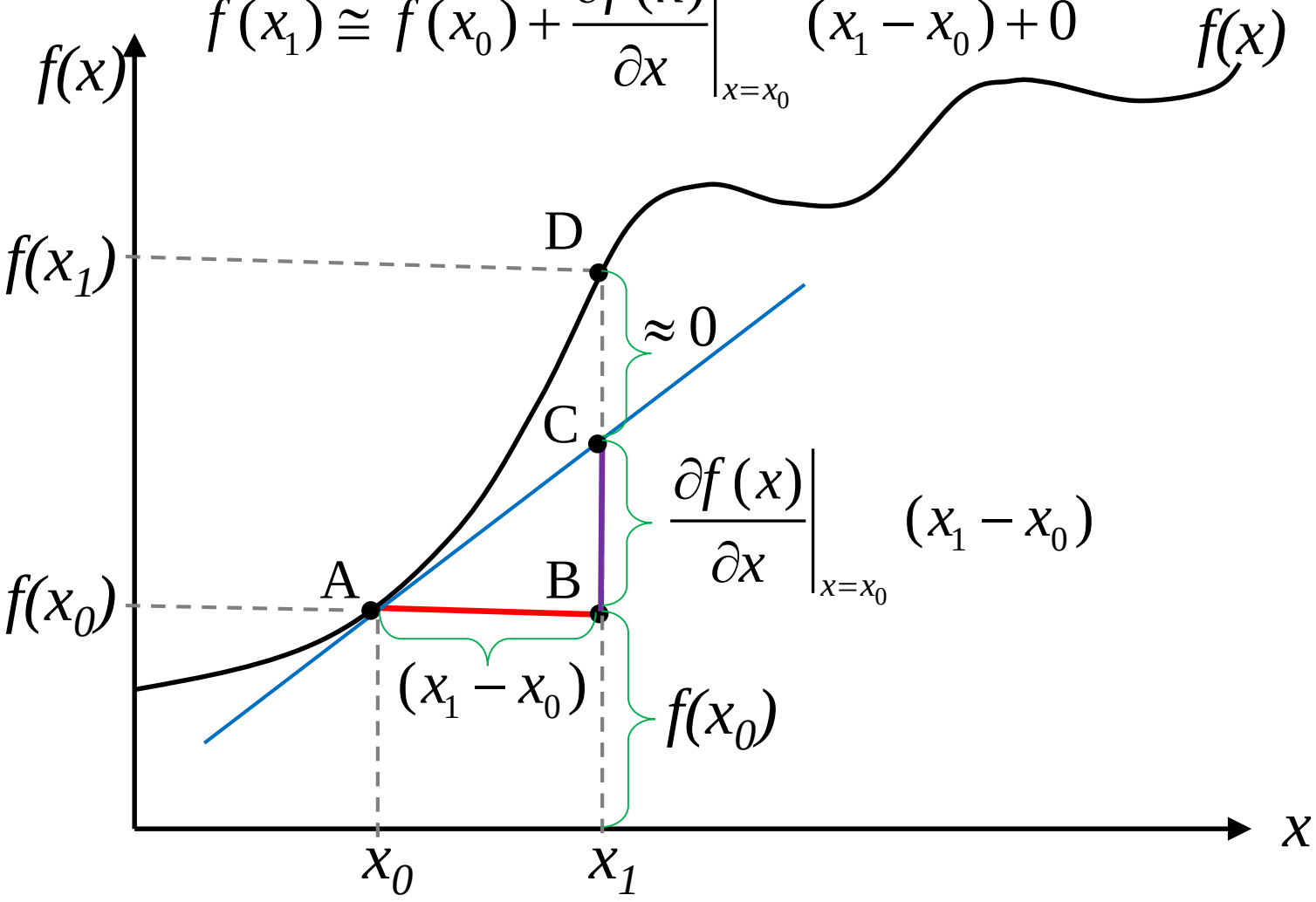
$f(x) \approx L(x)$ is standard linear approximation of f at x_0

The linearization is the equation of the tangent line.

Maximum Likelihood Computation

Linear Approximation of $f(x)$ at Point x_0

$$f(x_1) \cong f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0} (x_1 - x_0) + 0$$



Maximum Likelihood Computation

Taylor Linear Approximation

Linearize of log-likelihood function:

$$l(\theta) \approx l(\theta_0) + l'(\theta_0)(\theta - \theta_0)$$

where:

θ_0 is initial value of θ

$l'(\theta_0)$ is first derivative of $l(\theta)$ with respect to θ at point θ_0

$$l'(\theta_0) = \left.\frac{\partial l(\theta)}{\partial \theta}\right|_{\theta=\theta_0}$$

Maximum Likelihood Computation

Maximum Likelihood Estimation

Numerical Aspects of Optimization

Non-linear Optimization:

Gradient:

$$G = \frac{\partial l(\theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial l(\theta)}{\partial \theta_1} \\ \frac{\partial l(\theta)}{\partial \theta_2} \\ \vdots \\ \frac{\partial l(\theta)}{\partial \theta_k} \end{bmatrix}$$

Hessian:

$$H = \frac{\partial^2 l(\theta)}{\partial \theta \partial \theta'} = \begin{bmatrix} \frac{\partial^2 l(\theta)}{\partial \theta_1 \partial \theta_1} & \frac{\partial^2 l(\theta)}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 l(\theta)}{\partial \theta_1 \partial \theta_k} \\ \frac{\partial^2 l(\theta)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 l(\theta)}{\partial \theta_2 \partial \theta_2} & \cdots & \frac{\partial^2 l(\theta)}{\partial \theta_2 \partial \theta_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 l(\theta)}{\partial \theta_k \partial \theta_1} & \frac{\partial^2 l(\theta)}{\partial \theta_k \partial \theta_2} & \cdots & \frac{\partial^2 l(\theta)}{\partial \theta_k \partial \theta_k} \end{bmatrix}$$

Maximum Likelihood Computation

Maximum Likelihood Estimation

To maximize the log-likelihood function

FOC: $$G(\theta) = \frac{\partial l(\theta)}{\partial \theta} = 0$$

Linearized: $$f(x_1) \cong f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0} (x_1 - x_0)$$

$$G(\theta) \cong G(\theta_0) + \left.\frac{\partial G(\theta)}{\partial \theta}\right|_{\theta=\theta_0} (\theta - \theta_0)$$

$$G(\theta) \cong G(\theta_0) + H(\theta_0)(\theta - \theta_0)$$

Maximum Likelihood Computation

First-order condition: $G(\hat{\theta}) = 0$

Linearized around given value θ_0

$$G(\hat{\theta}) \cong G(\hat{\theta}_0) + H(\hat{\theta}_0)(\hat{\theta} - \hat{\theta}_0) = 0$$

Then, $G(\hat{\theta}_0) = H(\hat{\theta}_0)\hat{\theta} - H(\hat{\theta}_0)\hat{\theta}_0$

$$H(\hat{\theta}_0)\hat{\theta} = H(\hat{\theta}_0)\hat{\theta}_0 - G(\hat{\theta}_0)$$

$$\hat{\theta} = H(\hat{\theta}_0)^{-1} H(\hat{\theta}_0)\hat{\theta}_0 - H(\hat{\theta}_0)^{-1} G(\hat{\theta}_0)$$

$$\hat{\theta} = \hat{\theta}_0 - H(\hat{\theta}_0)^{-1} G(\hat{\theta}_0)$$

Newton-Raphson Algorithm

$$\hat{\theta}_{t+1} = \hat{\theta}_t - H(\hat{\theta}_t)^{-1} G(\hat{\theta}_t)$$

ML Computation – Algorithm

$$\hat{\theta}_{t+1} = \hat{\theta}_t - \Delta_t$$

Methods that employ line searches differ according to the choice of δ and methods of approximating $\lambda*$

Methods of choosing d mostly set up by letting $\lambda \geq 0$

Newton-Raphson $\Delta_t = H(\hat{\theta}_t)^{-1} G(\hat{\theta}_t)$

Quadratic Hill-climbing (Goldfeld-Quandt) Methods

$$\Delta = \tilde{H}^{-1} G(\hat{\theta}_t) \quad \text{where} \quad \tilde{H} = H(\hat{\theta}_t) + \gamma I$$

Newton $\Delta = K^{-1}(\hat{\theta}_t) G(\hat{\theta}_t)$ where $K = (GG') + H(\hat{\theta}_t)$

Guass-Newton or BHHH $\Delta = (GG')^{-1} G(\hat{\theta}_t)$

Marquardt $\Delta = (GG' + \gamma I)^{-1} G(\hat{\theta}_t)$

Maximum Likelihood Computation

Issues of Concern:

1. Initial Value – Global max vs Local max
2. Algorithm
3. Convergence Value – Default STATA = 10^{-13}
4. Maximum Iterative Times
   - Convergence achieved vs not achieved

ML Properties

Maximum Likelihood Estimation

ML in Linear Model

$$y = X\beta + \varepsilon, \qquad \varepsilon \sim N(0, \sigma^2 I)$$

Loglikelihood function:

$$l(\beta, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)$$

Maximize loglikelihood function by:

$$\frac{\partial l}{\partial \beta} = \frac{1}{\sigma^2} X'(y - X\beta) = 0$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(y - X\beta)'(y - X\beta) = 0$$

ML Properties

Maximum Likelihood Estimation

ML in Linear Model

$$\hat{\beta}_{ML} = (X'X)^{-1}X'y = \hat{\beta}_{OLS}$$

$$\hat{s}^2_{ML} = \frac{1}{n}(y - X\hat{\beta})'(y - X\hat{\beta}) = \frac{n-k}{n}\hat{s}^2_{OLS}$$

By assuming normality assumption ML and OLS provide the same results – unbiased β, but not s^2 -- only when $n \to \infty$

ML in Non-linear Regression Models

Assuming normality – NLS = ML

ML Properties

Asymptotic Properties

Asymptotic Distribution of ML Estimators

$$\sqrt{n}(\hat{\theta}_{ML} - \hat{\theta}_0) \xrightarrow{d} N(0, I_0^{-1})$$

Information Matrix:

$$I_n(\hat{\theta}_0) = E\left[\frac{\partial l}{\partial \theta}\frac{\partial l}{\partial \theta'}\right] = -E\left[\frac{\partial^2 l}{\partial \theta \partial \theta'}\right]$$

ML Properties

Asymptotic Properties

Approximate Distribution for Finite Samples

$$\hat{\theta}_{ML} \approx N\left(\theta_0, I_n^{-1}(\hat{\theta}_{ML})\right)$$

Information Matrix – Second order cond.:

$$\frac{\partial^2 l}{\partial \beta \partial \beta'} = -\frac{1}{\sigma^2} X'X$$

$$\frac{\partial^2 l}{\partial \beta \partial \sigma^2} = -\frac{1}{\sigma^4} X'(y - X\beta)$$

$$\frac{\partial^2 l}{\partial \sigma^2 \partial \sigma^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}(y - X\beta)'(y - X\beta)$$

ML Properties

Asymptotic Properties

Approximate Distribution for Finite Samples

Follows independent assumption:

$$I_n(\hat{\theta}_0) = \begin{pmatrix} \frac{1}{\sigma^2} X'X & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

Follows stability assumption:

$$I_n(\hat{\theta}_0) = \begin{pmatrix} \frac{1}{\sigma^2} Q & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}$$

ML Properties

Summary of Computations in ML

Step 1: Formulate the log-likelihood.

Step 2: Maximize the log-likelihood.

Step 3: Asymptotic tests.

ML Inference

Individual Test – *Z*-test

In OLS case, individual test is performed by using t-test because the estimated parameters are t-distributed.

In MLE case, distribution of the estimated parameters are not always t-distributed depending on distribution of the regression model. Thus, individual test can be performed by using *Z-test:* $H_0: \beta_i = 0$

$$z - test \approx \frac{\hat{\beta}_i}{s_{\hat{\beta}_i}} \sim N(0,1)$$

ML Inference

Likelihood Ratio (LR) Test

Based on the loss of log-likelihood that results if the restrictions are imposed.

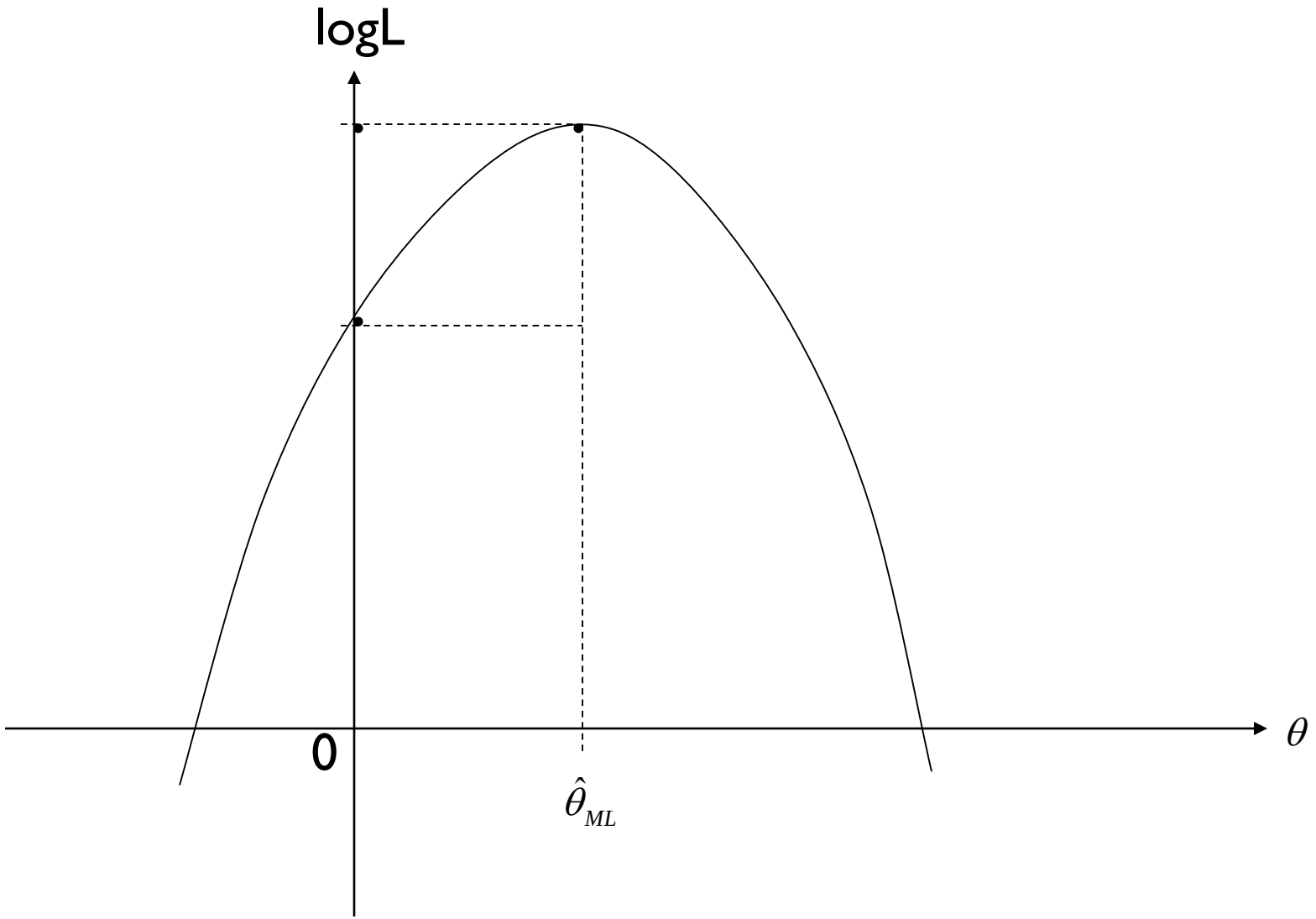
$$LR = 2\log(L(\hat{\theta}_1)) - 2\log(L(\hat{\theta}_0)) = 2l(\hat{\theta}_1) - 2l(\hat{\theta}_0)$$

H_0: $\theta = 0$

$$LR \xrightarrow{d} \chi^2(k-1)$$

ML Inference

Likelihood Ratio (LR) Test



ML Inference

Wald Test

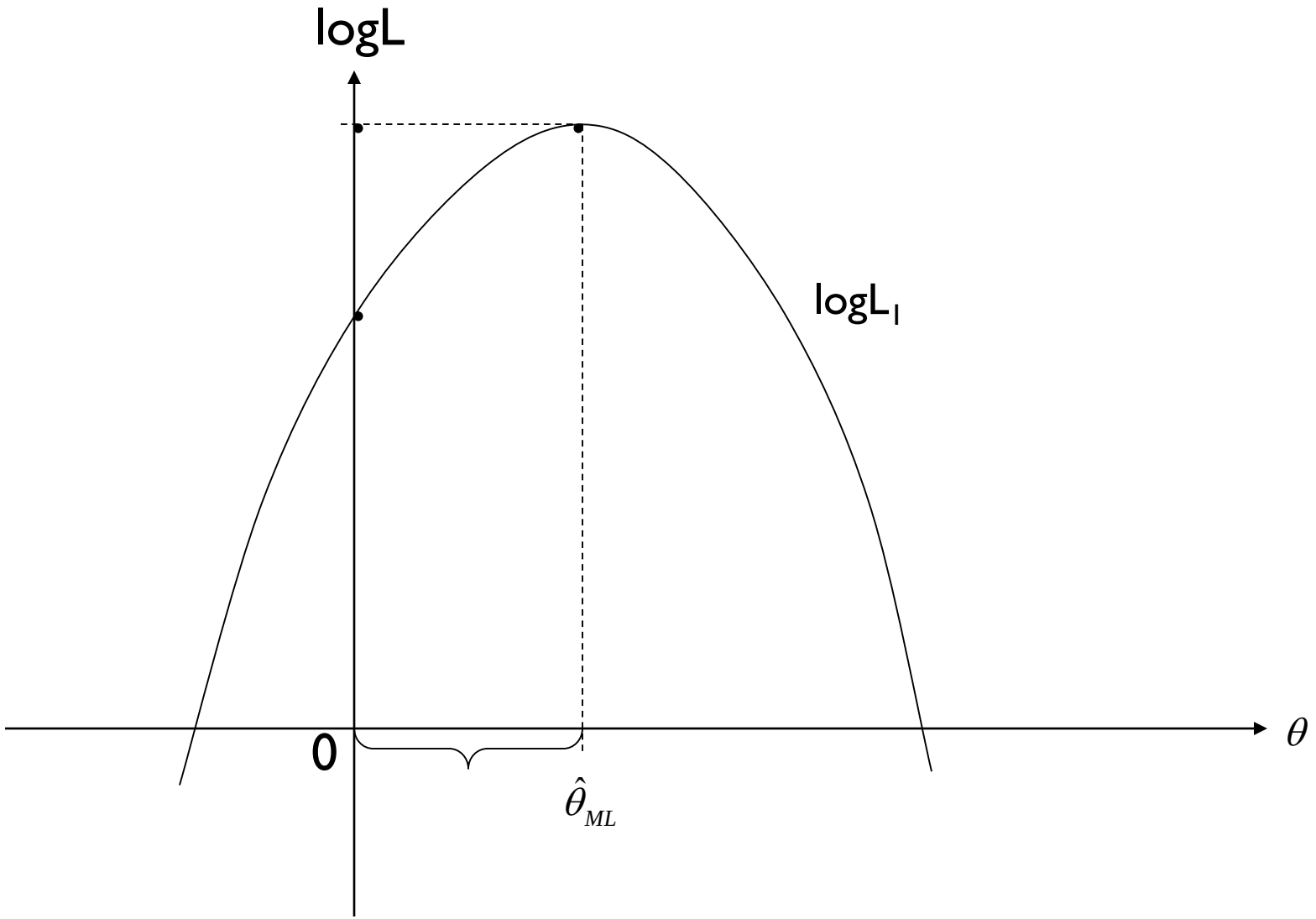
Based on unrestricted model alone.

$H_0: \theta = 0$

$$W = \hat{\theta}_1^2 \left(-\frac{d^2 l}{d\theta^2} \right) \approx \left(\frac{\hat{\theta}_1}{s_{\hat{\theta}_1}} \right) \approx \chi^2 (k-1)$$

ML Inference

Wald Test



ML Inference

Lagrange Multiplier (LM) Test

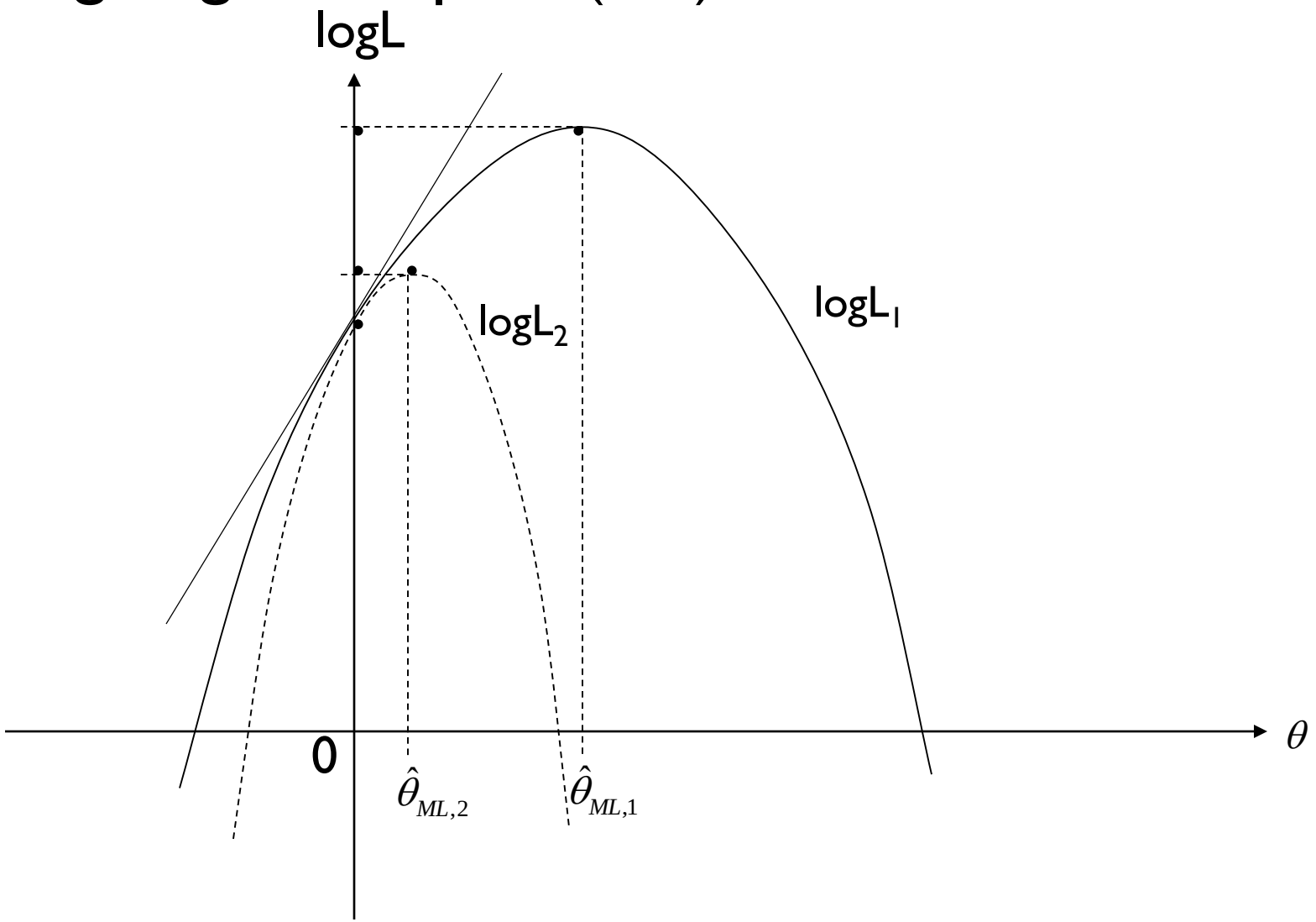
Score test considers whether the gradient (also called the 'score') of the unrestricted likelihood function is sufficiently close to zero at the restricted estimate θ.

$$H_0: \theta = 0$$

$$LM = \frac{(\partial l/\partial\theta)^2}{-\partial^2 l/\partial\theta^2} = \left(\frac{\partial l}{\partial\theta}\right)' \left(-E\left[\frac{\partial^2 l}{\partial\theta\partial\theta'}\right]\right)^{-1} \left(\frac{\partial l}{\partial\theta}\right) \approx \chi^2(k-1)$$

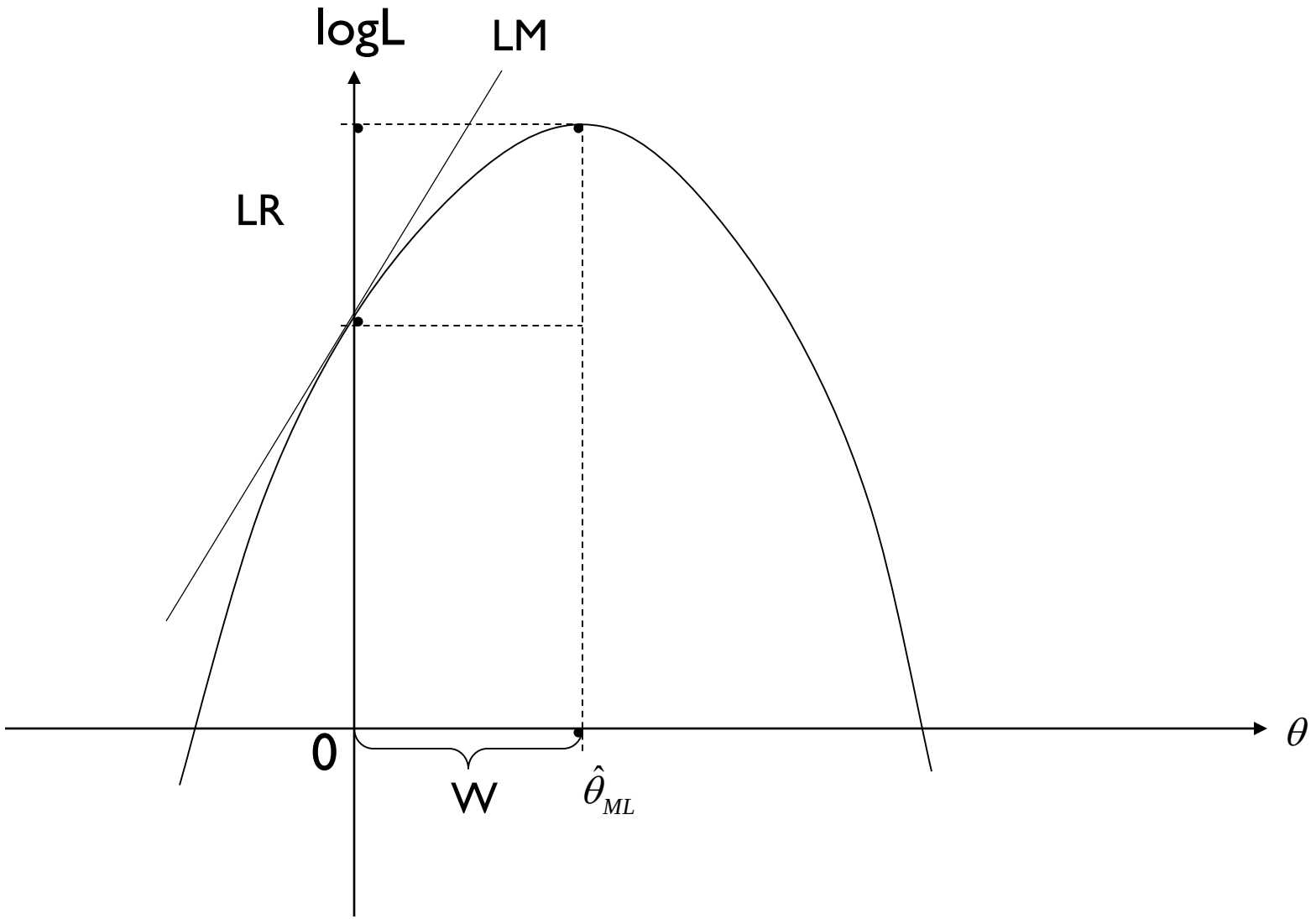
ML Inference

Lagrange Multiplier (LM) Test



ML Inference

Comparison of Three Tests



ML Inference

Comparison of Three Tests

	LR	Wald	LM
Estimated models	2 models	1 Unrestricted	1 Restricted
Advantage	Optimal power	Restricted model is complicated	Simple computations
Disadvantage	Needs 2 optimizations	Test depends on parameterization	Power may be small

$$LM \leq LR \leq W$$