Chapter 5 Inverses

Definition 5.1 Let $\mathbf{A}$ be a square matrix in $\mathbb{R}^{n\times n}$. The ***inverse of matrix*** $\mathbf{A}$, denoted by $\mathbf{A}^{-1} \in \mathbb{R}^{n\times n}$, is matrix with the property that,

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}$ is called ***invertible***.

Example Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$. Then $\mathbf{B}$ is an inverse of $\mathbf{A}$ because,

$$\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

Thus, we can write $\mathbf{B} = \mathbf{A}^{-1}$, and equivalently $\mathbf{A} = \mathbf{B}^{-1}$.

That is, from the definition of inverse, if matrix $\mathbf{B}$ is an inverse of matrix $\mathbf{A}$, then matrix $\mathbf{A}$ is also an inverse of matrix $\mathbf{B}$.

In showing that matrix $\mathbf{B}$ is an inverse of matrix $\mathbf{A}$, it will be shown in the next theorem that it is sufficient to show just either $\mathbf{BA} = \mathbf{I}$, or $\mathbf{AB} = \mathbf{I}$. To show this, we need the following definitions of left and right inverses.

Definition 5.2 Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be square matrices in $\mathbb{R}^{n\times n}$. Then the matrix $\mathbf{B}$ is a ***left inverse*** of $\mathbf{A}$ if $\mathbf{BA} = \mathbf{I}$, and the matrix $\mathbf{C}$ is a ***right inverse*** of $\mathbf{A}$ if $\mathbf{AC} = \mathbf{I}$.

Theorem 5.1 Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be square matrices in $\mathbb{R}^{n\times n}$. If the matrix $\mathbf{B}$ is a ***left inverse*** of $\mathbf{A}$ while the matrix $\mathbf{C}$ is a ***right inverse*** of $\mathbf{A}$, then $\mathbf{B} = \mathbf{C} = \mathbf{A}^{-1}$.

Proof By definition of inverse, it is sufficient to show just that The picture can't be displayed.. By the assumption,

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.\square$$

By this theorem, we also conclude that if $\mathbf{A}$ has an inverse, this inverse is the only inverse of $\mathbf{A}$.

Corollary 5.1 The inverse of matrix $\mathbf{A}$, if exists, is unique.

Proof Directly by Theorem 5.1.$\square$

Example As we have seen in the proof of Theorem 3.5, for any elementary matrix $\mathbf{E}$, there exists another elementary matrix $\mathbf{F}$ such that $\mathbf{FE} = \mathbf{I}$. Then by Theorem 5.1, we also have $\mathbf{EF} = \mathbf{I}$, and so $\mathbf{E}^{-1} = \mathbf{F}$ and $\mathbf{F}^{-1} = \mathbf{E}$.

Problem Leon [1994], # 18 a, 19, page 62.

19. Let $\mathbf{A}$ be a nonsingular matrix and let $\mathbf{B}$ be an $n \times n$ matrix. Show that the reduced row echelon form of $[\mathbf{A} \quad \mathbf{B}]$ is $[\mathbf{I} \quad \mathbf{C}]$, where $\mathbf{C} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

5.1 Properties of Inverse

We have one property of the inverse by Theorem 5.1 that the inverse of a matrix, if exists, is unique. The following two theorems are additional properties of inverse.

Theorem 5.2 Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices in $\mathbb{R}^{n\times n}$. If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Proof By Theorem 5.1, it is sufficient to check if $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$. This is left as an exercise.❑

Corollary 5.2

a. If $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_k \in \mathbb{R}^{n\times n}$ are invertible, then

$$(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1} \cdots \mathbf{A}_1^{-1}$$

b) If $\mathbf{A}$ is invertible, then $(\mathbf{A}^m)^{-1} = (\mathbf{A}^{-1})^m$

Proof By Theorem 5.2 and simple induction.❑

Theorem 5.3 Let $\mathbf{A}$ be invertible. Then,

a) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
b) $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$.
c) $|\mathbf{A}^{-1}| = \frac{\mathbf{1}}{|\mathbf{A}|}$.

Proof **a)** Since $\mathbf{A}^{-1}(\mathbf{A}^{-1})^{-1} = \mathbf{I}$, premultiplying both sides by $\mathbf{A}$, we have,

$$\mathbf{AA}^{-1}(\mathbf{A}^{-1})^{-1} = \mathbf{AI} = \mathbf{A}$$
$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}.$$

b. Since $\mathbf{I}^{\mathrm{T}} = (\mathbf{A}^{-1}\mathbf{A})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{I}$, premultiplying both sides of the last equality by $(\mathbf{A}^{\mathrm{T}})^{-1}$ and we have the required result.

c. $1 = |\mathbf{I}| = |\mathbf{A}^{-1}\mathbf{A}| = |\mathbf{A}^{-1}||\mathbf{A}|$.❑

Problem Show that $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$.

Problem Leon [1994], #21, page 62. If a symmetric matrix **A** is invertible, then its inverse is also symmetric.

Problem Fraleigh & Beauregard [1995], page 86, #38.

38. Let **A** be an invertible square matrix. Recalling Lemma 4.1 (p.29 of this lecture notes) and that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, answer the following questions:
    a. If two rows of **A** are interchanged, how does the inverse of the resulting matrix compare with $\mathbf{A}^{-1}$?
    b. Answer the question in part **(a)** if, instead, a row of **A** is multiplied by a nonzero scalar *r*.
    c. Answer the question in part **(a)** if, instead, r times the i^{th} row of **A** is added to the j^{th} row.

5.2 Computation of Inverse

What we have discussed so far are the properties of the inverse of a matrix. We will now find such an inverse matrix, and more importantly, what are the conditions that guarantee the existence of the inverse of a matrix.

If we have a computational algorithm to compute the inverse of a matrix, one of its obvious use is to solve systems of linear equations. If $\mathbf{Ax} = \mathbf{b}$ is a system of n linear equations of n variables, then **A** is square. Suppose $\mathbf{A}^{-1}$ exists, then the solution **x** is given by,

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{Ax} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}. \end{aligned}$$

Therefore, the calculation of $\mathbf{A}^{-1}$ is equivalent to solving $\mathbf{Ax} = \mathbf{b}$.

The two usual ways of computing $\mathbf{A}^{-1}$ are by using elementary row operations and by the adjoint matrix.

Theorem 5.4 Let **A** be a matrix in $\mathbb{R}^{n\times n}$. If there exists a series of k elementary row operations, each represented by premultiplying **A** by an elementary matrix $\mathbf{E}_i, i = 1,2,\dots,k$, such that when performed sequentially on **A** we have $\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1\mathbf{A} = \mathbf{I}$, then $\mathbf{A}^{-1} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1$.

Proof Let $\mathbf{B} = \mathbf{E}_k \mathbf{E}_{k-1} \cdots \mathbf{E}_1$. Then $\mathbf{B}$ is the inverse of $\mathbf{A}$ by Theorem 5.1.❑

In practice, we write the augmented matrix $\widehat{\mathbf{A}} = [\mathbf{A} \quad \mathbf{I}]$, where the first n columns are just columns of $\mathbf{A}$ and the last n columns constitute an identity matrix. Then perform the elementary row operations on $\widehat{\mathbf{A}}$ until the first n columns becomes an identity matrix.

Example Find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. We have,

$$\begin{aligned} \widehat{\mathbf{A}} = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] &\xRightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right] \\ &\xRightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -0.5 \end{array}\right] \\ &\xRightarrow{R_1 - 3R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1.5 \\ 0 & 1 & 1 & -0.5 \end{array}\right] \end{aligned}$$

The inverse is $\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix}$.

Problem Can we find $\mathbf{A}^{-1}$ by performing elementary column operations on $\widehat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix}$ until we obtain $\begin{bmatrix} \mathbf{I} \\ \mathbf{B} \end{bmatrix}$ and say $\mathbf{B} = \mathbf{A}^{-1}$?

5.2.2 Computing Inverses by Adjoint Matrix

Definition 5.3 Let $\mathbf{A}$ be a matrix in $\mathbb{R}^{n \times n}$. The ***adjoint of*** $\mathbf{A}$, denoted by $\boldsymbol{adj}\ \mathbf{A}$, is a matrix in $\mathbb{R}^{n \times n}$ and given by

$$\begin{aligned} adj\ \mathbf{A} &= [\Delta_{ij}]^{\mathrm{T}} \\ &= \begin{bmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{n1} & \Delta_{n2} & \cdots & \Delta_{nn} \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

where Δ_{ij} is the $(i,j)^{\text{th}}$ cofactor of $\mathbf{A}$.

Theorem 5.5 Let $\mathbf{A}$ be a nonsingular matrix. The inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj\ \mathbf{A}$$

Proof By Theorem 4.1, the determinant of $\mathbf{A}$ can be written as,

$$|\mathbf{A}| = \begin{cases} \sum_{j=1}^{n} a_{ij}\Delta_{ij}, & i = 1,2,\dots,n \\ \sum_{i=1}^{n} a_{ij}\Delta_{ij}, & j = 1,2,\dots,n \end{cases}$$

Now by the form of the inverse $\mathbf{A}^{-1}$ as given in the theorem,

$$\begin{aligned}
&\mathbf{A}\mathbf{A}^{-1} \\
&= \frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{n1} & \Delta_{n2} & \cdots & \Delta_{nn} \end{bmatrix}^{\mathrm{T}} \\
&= \frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{1n} & \Delta_{2n} & \cdots & \Delta_{nn} \end{bmatrix} \\
&= \frac{1}{|\mathbf{A}|}\begin{bmatrix} \sum_{j=1}^{n} a_{1j}\Delta_{1j} & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{n} a_{2j}\Delta_{2j} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{n} a_{nj}\Delta_{nj} \end{bmatrix} \\
&= \mathbf{I}.
\end{aligned}$$

The last two equalities are due to the properties of the determinant stated in Theorem 4.1 and 4.2.❑

Problem Verify that

$$\sum_{j=1}^{n} a_{ij}\Delta_{kj} = \begin{cases} |\mathbf{A}|, & i = j, \\ 0, & i \neq j. \end{cases}$$

Problem Fraleigh & Beauregard [1995], page 272, #35 (*g*)-(*i*), 36-39.

35. Let **A** be a square matrix. Mark each of the following True or False.

_____**g.** The product of a square matrix and its adjoint is the identity matrix.

_____**h.** The product of a square matrix and its adjoint is equal to some scalar times the identity matrix.

_____**i.** The transpose of the adjoint of **A** is the matrix of cofactors of **A**.

36. Prove that the inverse of a nonsingular upper-triangular matrix is upper triangular.
37. Prove that a square matrix is invertible if and only if its adjoint is an invertible matrix.
38. Let **A** be an matrix. Prove that $|adj\,\mathbf{A}| = (|\mathbf{A}|)^{n-1}$.
39. Let **A** be an invertible matrix. Using Exercises 37 and 38, prove that $adj\,(adj\,\mathbf{A}) = (|\mathbf{A}|)^{n-2}\mathbf{A}$.

Problem Fraleigh & Beauregard [1995], page 85, #26 and 30.

26. Let **A** be a matrix such that $\mathbf{A}^2$ is invertible. Prove that **A** is invertible.
30. A square matrix **A** is said to be idempotent if $\mathbf{A}^2 = \mathbf{A}$.
    a. Give an example of an idempotent matrix other than **0** and **I**.
    b. Show that, if a matrix **A** is both idempotent and invertible, then $\mathbf{A} = \mathbf{I}$.

5.3 Inverse and Determinants

Theorem 5.6 A matrix **A** in $\mathbb{R}^{n\times n}$ is ***invertible*** if, and only if, **A** is nonsingular.

Proof We have to show that $\mathbf{A}^{-1}$ exists if, and only if, $|\mathbf{A}| \neq 0$. Suppose $|\mathbf{A}| \neq 0$, then $\mathbf{A}^{-1}$ can be computed using the adjoint of **A**. Therefore $\mathbf{A}^{-1}$ exists.

Now suppose $\mathbf{A}^{-1}$ exists but $|\mathbf{A}| = 0$. Since $\mathbf{A}^{-1}$ exists we can compute its determinant $|\mathbf{A}^{-1}|$ to be some real number. However, by Theorem 5.3 part **(c)**, $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$, which is undefined as $|\mathbf{A}| = 0$. This is a contradiction. So $|\mathbf{A}| \neq 0$.❑

Similar to Corollary 4.3 in the previous chapter, we can identify a system of n linear equations with n variables as having unique solution by the coefficient matrix **A** being full rank, nonsingular or invertible.

Corollary 5.3 Let **A** be a matrix in $\mathbb{R}^{n\times n}$. The following statements are equivalent:

1. $\mathbf{A}^{-1}$ exists
2. $|\mathbf{A}| \neq 0$, i.e., nonsingular
3. $rank\,\mathbf{A} = rank\,\mathbf{A}^{\mathrm{T}} = n$.
4. $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any given vector **b**.

Proof By Theorem 5.6 and Corollary 4.3.❑

Problem Write a corollary summarizing results when $\mathbf{A}^{-1}$ does not exist.

Problem Fraleigh & Beauregard [1995], page 85, #25.

25. **a)** If $\mathbf{A}$ is invertible, is $\mathbf{A}+\mathbf{A}^{\mathrm{T}}$ always invertible?

b) If $\mathbf{A}$ is invertible, is $\mathbf{A}+\mathbf{A}$ always invertible?

Problem Johnson, Riess & Arnold [1998], page 103, #67, 68, 70, and 71.

67. Suppose that $\mathbf{AB}=\mathbf{0}$, where $\mathbf{A}$ is nonsingular. Prove that $\mathbf{B}=\mathbf{0}$.
68. Let $\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ be matrices such that $\mathbf{A}$ is nonsingular and $\mathbf{AB}=\mathbf{AC}$. Prove that $\mathbf{B}=\mathbf{C}$.
70. Let $\mathbf{A}$ and $\mathbf{B}$ be $n\times n$ nonsingular matrices. Show that $\mathbf{AB}$ is nonsingular.
71. Is it true that if $\mathbf{AB}$ is nonsingular, then both $\mathbf{A}$ and $\mathbf{B}$ must be nonsingular?

Problem Simon & Blume [1994], page 739, # 26, 28.

a. Prove that if the entries of $\mathbf{A}$ are all integers and if $|\mathbf{A}|=\pm 1$, then the entries of $\mathbf{A}^{-1}$ are also integers.
b. Use Theorem 5.3 (c) to show that if the entries of $\mathbf{A}$ and $\mathbf{A}^{-1}$ are all integers, then $|\mathbf{A}|=\pm 1$.

5.4 Cramer's Rule

The system of n linear equations with n variables can be solved by using the so-called Cramer's rule, which use the properties of the inverse and determinant.

Theorem 5.7 (Cramer's Rule) Given a system of n linear equations with n variables $\mathbf{Ax}=\mathbf{b}$, if $\mathbf{A}$ is nonsingular, then the solution to the system of linear equations are uniquely given by,

$$x_j=\frac{|\mathbf{a}_{\cdot 1} \quad \cdots \quad \mathbf{a}_{\cdot j-1} \quad \mathbf{b} \quad \mathbf{a}_{\cdot j+1} \quad \cdots \quad \mathbf{a}_{\cdot 1}|}{|\mathbf{A}|}, j=1,2,\dots,n,$$

where the numerator of the right hand side of the equation is the determinant of matrix $\mathbf{A}$, but with its j^{th} column replaced by the right hand side vector $\mathbf{b}$.

Proof Since $\mathbf{A}^{-1}$ exists, the solution is given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, which by the calculation of $\mathbf{A}^{-1}$ by the adjoint matrix of $\mathbf{A}$ as stated in Theorem 5.5,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{1n} & \Delta_{2n} & \cdots & \Delta_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$
$$= \frac{1}{|\mathbf{A}|}\begin{bmatrix} \sum_{i=1}^{n} \Delta_{i1} b_i \\ \sum_{i=1}^{n} \Delta_{i2} b_i \\ \vdots \\ \sum_{i=1}^{n} \Delta_{in} b_i \end{bmatrix}.$$

The theorem follows since each of the terms $\sum_{i=1}^{n} \Delta_{ij} b_i$, $j = 1,2,\dots,n$, is the determinant of the matrix $|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j-1} \ \ \mathbf{b} \ \ \mathbf{a}_{\cdot j+1} \ \cdots \ \mathbf{a}_{\cdot 1}|$ as expanded along the j^{th} column $\mathbf{b}$. Then x_j is uniquely determined as given in the statement of the theorem.❑

Problem Fraleigh & Beauregard [1995], page 272, #34. Find the unique solution (assuming that it exists) of the system of equations represented by the augmented matrix.

$$\left[\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & 3b_1 \\ a_2 & b_2 & c_2 & d_2 & 3b_2 \\ a_3 & b_3 & c_3 & d_3 & 3b_3 \\ a_4 & b_4 & c_4 & d_4 & 3b_4 \end{array}\right]$$