Fundamentals of Mathematical Proofs: II

TU152: Fundamental Mathematics

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Method of Generalizing from the Generic Particular

In order to use the *method of generalizing from the generic particular* for

proving

$$\forall x \in D, P(x)$$

or disproving

$$\exists x \in D, Q(x),$$

it is helpful to use the following three steps:

1. Restate the claim in a **formal** way.
2. Specify the **starting point**.
3. Identify **the conclusion to be shown**.

Method of Generalizing from the Generic Particular

Example: (Disproving an Existential Statement)
Disprove the following statement:

There is a positive integer n such that $n^2 + 3n + 2$ is prime.

Rational Numbers

Definition

A real number r is **rational** if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator. A real number that is not rational is **irrational**. More formally, if r is a real number, then

$$r \text{ is rational } \Leftrightarrow \exists \text{ integers } a \text{ and } b \text{ such that } r = \frac{a}{b}, b \neq 0.$$

Note: The word *rational* contains the word *ratio*, which is another word for *quotient*. A *rational number* can be written as a *ratio* of integers.

Example: Determine whether the following numbers are rational or irrational.

1. 0
2. $10/3$
3. $-\frac{3}{47}$
4. 0.1234
5. $0.12121212\ldots$ (where the digits 12 are assumed to repeat forever)

Rational Numbers

Example: Prove the following theorem.

Theorem

The sum of any two rational numbers is rational.

Rational Numbers

A **corollary** is a statement whose truth can be immediately deduced from a theorem that has already been proved.

Example:

Corollary

The double of a rational number is rational.

More Methods of Proof

Vacuous Proof

A **vacuous proof** is a proof of an implication p ! q in which it is shown that p is false.

Example Use the method of vacuous proof to show that if $x \in \emptyset$, then David is playing soccer.
Answer:

Trivial Proof

A **trivial proof** of an implication $p \to q$ is one in which q is shown to be true without any reference to p.

Example Use the method of trivial proof to show that if n is an even integer then n is divisible by 1.
Answer:

More Methods of Proof: Method of proof by cases

Method of Proof by Cases

The method of proof by cases is a direct method of proving the conditional statement

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \to q.$$

The method consists of proving the conditional statements

$$p_1 \to q, p_2 \to q, \ldots, p_n \to q.$$

To prove a statement of the form

"If $A1$ or $A2$ or … or A_n, then C"

we have to prove all of the following:

- If A_1, then C,
- If A_2, then C,

  ⋮

  ⋮
- If A_n, then C.

This process shows that C is true regardless of which of $A_1, A_2, \ldots, A_n$ happens to be the case.

Method of Proof by Cases

Example: Proof the following statement.

If n is a positive integer, then $n^3 + n$ is even.

Method of Proof by Cases:Absolute value example

Definition (Absolute Value)

For any real number x, the absolute value of x, denoted $|x|$, is defined as follows:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Example: Use the proof by cases to prove the triangle inequality:
For all real numbers x and y,

$$|x + y| \leq |x| + |y|.$$

Floor $\lfloor x \rfloor$ and Ceiling $\lceil x \rceil$

Definition (Floor)

Given any real number x, the floor of x, denoted $\lfloor x \rfloor$, is defined as:

$$\lfloor x \rfloor = n, \quad n \text{ is an integer such that } n \leq x < n+1.$$

Symbolically, if x is a real number,

$$\boxed{\lfloor x \rfloor = n \quad \Leftrightarrow \quad n \in \mathbb{Z}, \;\; n \leq x < n+1.}$$

Definition (Ceiling)

Given any real number x, the ceiling of x, denoted $\lceil x \rceil$, is defined as:

$$\lceil x \rceil = n, \quad n \text{ is an integer such that } n-1 < x \leq n.$$

Symbolically, if x is a real number,

$$\boxed{\lceil x \rceil = n \quad \Leftrightarrow \quad n \in \mathbb{Z}, \;\; n-1 < x \leq n.}$$

Floor $\lfloor x \rfloor$ and Ceiling $\lceil x \rceil$

Example: Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ of the following values of x.

1. $x = 37.999$
2. $x = 11$
3. $x = 0.6$
4. $x = -\frac{57}{2}$
5. $x = -14.001$

Floor $\lfloor x \rfloor$ and Ceiling $\lceil x \rceil$

Example: Use the proof by a counterexample to show that the statement

$$\text{“}\forall x, y \in \mathbb{R}, \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor\text{”}$$

is **false**.

Answer:

Floor $\lfloor x \rfloor$ and Ceiling $\lceil x \rceil$

Example: Use the method of proof by cases to prove the following statement.

Let n be an integer. Then

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}.$$