Mathematical Induction I

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer.

Suppose the following two statements are true:

1. $P(a)$ is true.

2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement " for all integers $n \geq a$, $P(n)$" is true.

The validity of proof by mathematical induction is generally taken as an axiom. That is why it is referred to as the principle of mathematical induction rather than as a theorem.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true."

To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

where k is any particular but arbitrarily chosen integer with $k \geq a$.
Then show that $P(k+1)$ is true.

Definition

If n and d are integers and $d \neq 0$ then n is divisible by d if, and only if, n equals d times some integer. The notation $d|n$ is read "d divides n." Symbolically, if n and d are integers and $d \neq 0$

$$d|n \Leftrightarrow \exists \text{ an integer } k \text{ such that } n = dk.$$

Instead of "n is divisible by d," we can say that

- n is a multiple of d, or
- d is a factor of n, or
- d is a divisor of n, or
- d divides n.

Definition Closed Form If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis or a summation symbol, we say that it is written in closed form.

Example Use mathematical induction to prove that for all integers $n \geq 1$.

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Example(Sum of a Geometric Sequence)
Use mathematical induction to prove that for any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Example Use mathematical induction to prove that $n < 2^n$ for all non-negative integers n.

Example Use mathematical induction to prove that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Example Use mathematical induction to prove that for all integers $n \geq 3$,

$$2n + 1 < 2^n.$$

Example Define a sequence $a_1, a_2, a_3, \ldots$ as follows.

$$\begin{aligned} a_1 &= 2 \\ a_k &= 5a_{k-1} \quad \text{for all integers } k \geq 2. \end{aligned}$$

1. Write the first four terms of the sequence.
2. It is claimed that for each integer $n \neq 1$, the value of the n-th term of the sequence is given by the formula $2 \cdot 5^{n-1}$. I.e.

$$a_n = 2 \cdot 5^{n-1}.$$

   Prove this claim.

Strong Mathematical Induction

Recall that the method of proof by mathematical induction is given by the following two main steps.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true."

To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

where k is any particular but arbitrarily chosen integer with $k \geq a$.
Then show that $P(k+1)$ is true.

Strong mathematical induction is similar to *ordinary mathematical induction* in that :

- it is a technique for establishing the truth of *a sequence of statements about integers*;
- it consists of a *basis step* and an *inductive step*.

However, the **strong mathematical induction** is different from *ordinary mathematical induction* as follows.

- The basis step of **strong mathematical induction** may contain proofs for several initial values,
- In the inductive step of **strong mathematical induction**, the truth of the predicate $P(n)$ is assumed not just for one value of n but for all values through k, and then the truth of $P(k+1)$ is proved.

Method of Proof by Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers

with $a \leq b$. Suppose two statements of the following steps are true:

Step 1 (basis step): Show that $P(a), P(a+1), \ldots$ and $P(b)$ are all true.

Step 2 (inductive step): Show that for any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k,

then $P(k+1)$ is true.

Then the statement for all integers $n \geq a$, $P(n)$ is true.

Note that the supposition that $P(i)$ is true for all integers i from a through k is called the **inductive hypothesis**. Another way to state the inductive hypothesis is to say that $P(a), P(a+1), ..., P(k)$ are all true.

Historical Notes *The first known use of mathematical induction was in the work of the 16th-century mathematician Francesco Maurolico (1494-1575). In his book Arithmeticorum Libri Duo, he presented a variety of properties of the integers together with proofs of these properties. He used induction to prove some of these properties, the first one being the proof that the sum of the first n odd integers is n^2. The first formal explanation of mathematical induction was presented by Augustus DeMorgan (1806-1871) in 1838. This is also the first time the term "induction" was used in this context.*

Example(Divisibility by a Prime)
Use strong mathematical induction to prove the following statement.

Any integer greater than 1 is divisible by a prime number.

Example(Proving a Property of a Sequence with Strong Induction)
Define a sequence $s_0, s_1, s_2, \ldots$ as follows:

$$s_0 = 0, \quad s_1 = 4, \quad s_k = 6a_{k-1} - 5a_{k-2} \quad \text{for all integers } k \geq 2.$$

1. Find the first four terms of this sequence.
2. It is claimed that for each integer $n \geq 0$, the value of the n-th term of the sequence can be given by the formula

$$s_n = 5^n - 1.$$

   Prove that this is true.

Example (Existence and Uniqueness of Binary Integer Representations)
Given any positive integer n, n has a unique representation in the form

$$n = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0,$$

where r is a nonnegative integer, $c_r = 1$, and $c_j = 1$ or 0 for all $j = 0, 1, 2, \ldots, r - 1$.

The Well-Ordering Principle for the Integers

> **Well-Ordering Principle for the Integers**
>
> Let S be a set of integers containing one or more integers all of which are greater than some fixed integer. Then S has a least element.

The well-ordering principle for the integers (or simply "well ordering principle") looks very different from both the ordinary and the strong principles of mathematical induction, but it can be shown that all three principles are equivalent. That is, if any one of the three is true, then so are both of the others.