Assignment 4 EE320 (Section Aj. Kittichai)

Due on Nov., 26th 2020

Instruction

1) All groups must attempt question 0.
2) Odd-numbered group must attempt all odd-numbered questions.
3) To submit your homework, write your filename as follow **hw4_Group_0x**. One point will be deducted if you don't follow the format of suggested filename.

Question 0: (Three-variable optimization)

Suppose that there are three firms. Each firm produces one single product that is imperfectly substitutable to the other two products, i.e. differentiated products. Each of the firm *strategically* competes with each other, and faces the following market demand equations given by,

$$Q_1 = 80 - 2P_1 + P_{23}$$

$$Q_2 = 80 - 2P_2 + P_{13}$$

$$Q_3 = 80 - 2P_3 + P_{12}$$

where P_{23} is the average of the prices charged by firms 2 and 3, P_{13} is the average of the prices charged by firms 2, P_{12} is the average of the prices charged by firms 1 and 2 [e.g., $P_{12} = 0.5(P_1 + P_2)$]. Suppose that each of the three firms, as indexed by j, has the cost function given by $C^j\left(Q_j\right) = c_j Q_j$.

Consider the following problems.

a) Set up the profit function of each individual firm, and derive the profit-maximizing condition of each of the three firms when each of them *optimally* sets for the level of price that maximizes its own profit. (**Hint:** *Derive the best response function of each firm, i.e. optimal contingent plan or reaction function.*)

b) Suppose that Based on (a), what would be the *equilibrium* level of price that each of the firms choose. Calculate the level of profit that each firm yields. (Hint: *Solve for the Bertrand equilibrium with differentiated product*)

c) How does the optimal price vary with respect to the size of marginal cost?

Now suppose that all the three firms agree to operate under a cartel (collusive) agreement. That is, they consider a joint pricing scheme that maximizes the joint profit function of the three firms combined. Consider the following problems.

d) Construct the joint profit function.

e) Given that $c_j = c,\ \forall j$. Calculate the level of optimal price setting that each of the firm will set under the joint profit function problem. Confirm your result with the second-order derivative test.

f) Is the Cartel agreement self-sustaining? Why?

Question 0: (Three-variable optimization)

Suppose that there are three firms. Each firm produces one single product that is imperfectly substitutable to the other two products, i.e. differentiated products. Each of the firm *strategically* competes with each other, and faces the following market demand equations given by,

$$Q_1 = 80 - 2P_1 + P_{23}$$

$G^r: 240 - 6P + 3P$

$$Q_2 = 80 - 2P_2 + P_{13}$$

$$Q_3 = 80 - 2P_3 + P_{12}$$

where P_{23} is the average of the prices charged by firms 2 and 3, P_{13} is the average of the prices charged by firms 2, P_{12} is the average of the prices charged by firms 1 and 2 [e.g., $P_{12} = 0.5(P_1 + P_2)$]. Suppose that each of the three firms, as indexed by j, has the cost function given by $C^j(Q_j) = c_j Q_j$.

Consider the following problems.

a) Set up the profit function of each individual firm, and derive the profit-maximizing condition of each of the three firms when each of them *optimally* sets for the level of price that maximizes its own profit. (**Hint:** *Derive the best response function of each firm, i.e. optimal contingent plan or reaction function.*)

$$\pi_1 = P_1(80 - 2P_1 + 0.5P_2 + 0.5P_3) - C_1(80 - 2P_1 + 0.5P_2 + 0.5P_3)$$

$$\frac{\partial \pi_1}{\partial P_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 2C_1 = 0$$

$$-4P_1 = -80 - 0.5P_2 - 0.5P_3 - 2C_1$$

$$\left.\begin{aligned} P_1 &= \frac{80 + 0.5P_2 + 0.5P_3 + 2C_1}{4} \\ P_2 &= \frac{80 + 0.5P_1 + 0.5P_3 + 2C_2}{4} \\ P_3 &= \frac{80 + 0.5P_1 + 0.5P_2 + 2C_3}{4} \end{aligned}\right\} \text{Since function of } Q_{1,2,3} \text{ is the same}$$

b) Suppose that Based on (a), what would be the *equilibrium* level of price that each of the firms choose. Calculate the level of profit that each firm yields. (Hint: *Solve for the Bertrand equilibrium with differentiated product*)

$$P_1 = \frac{80}{4} + \frac{0.5}{4}\left(\frac{80 + 0.5P_1 + 0.5P_3 + 2C_2}{4}\right) + \frac{0.5}{4}\left(\frac{80 + 0.5P_1 + 0.5P_2 + 2C_3}{4}\right) + \frac{2C_1}{4}$$

$$P_1 = 20 + \frac{40 + 0.25P_1 + 0.25P_3 + C_2}{16} + \frac{40 + 0.25P_1 + 0.25P_2 + C_3}{16} + \frac{2C_1}{4}$$

$$P_1 - \frac{0.5P_1}{16} = \frac{400 + 0.25P_2 + 0.25P_3 + C_2 + C_3 + 8C_1}{16}$$

$$P_1 = \frac{400 + 0.25P_2 + 0.25P_3 + C_2 + C_3 + 8C_1}{15.5}$$

$$P_2 = \frac{400 + 0.25P_1 + 0.25P_3 + C_1 + C_3 + 8C_2}{15.5}$$

$$P_3 = \frac{400 + 0.25P_1 + 0.25P_2 + C_1 + C_2 + 8C_3}{15.5}$$

Since function $Q_{1,2,3}$ is the same

c) How does the optimal price vary with respect to the size of marginal cost?

$$\frac{\Delta P_1, P_2, P_3}{\Delta C} = \frac{8}{15.5}$$

meaning that 1\$ increase in MC = 8/15.5 \$ increase in Price for all 3 firm

d) Construct the joint profit function.

$Q^T = Q_1 + Q_2 + Q_3 = 240 - 6P + 3P$

$\pi^T = 240P - 6P^2 + 3P^2 - C_1(80-P) - C_2(80-P) - C_3(80-P)$

e) Given that $c_j = c,\ \forall j$. Calculate the level of optimal price setting that each of the firm will set under the joint profit function problem. Confirm your result with the second-order derivative test.

$\pi^T = 240P - 3P^2 - 240C + 3CP$

$$\frac{\partial \pi^T}{\partial P} = 240 - 6P + 3C = 0$$

$$S.O.C = \frac{\partial \pi^{T2}}{\partial^2 P} = -6 < 0$$

$\therefore$ Hence function is concave so we have maximum profit

$$P = \frac{240 + 3C}{6}$$

(All 3 firm use same price under cartel)

f) Is the Cartel agreement self-sustaining? Why?

Yes, since the MC of all firm are the same, there would not be any firm with more power to exploit the market. If one firm decide to undercut price, all firm will follow and lead to price finally at MC. Hence giving no economic for every firm if they decide to cheat.

g) Revisit question (e) and assume that the marginal costs are varied with respect to firms. What would be the implication of the collusive equilibrium?

If MC varied too much, the collusive equilibrium would not be possible because one firm will have too much power over the other, and can lower theirs price to be a little below the others to drive them out.

Question 1:

Given the production function

$$Q = f(K, L) = K^{0.5}L^{0.5}$$

Suppose that the per unit input prices for K and L are \$20 and \$5, respectively.

a. If the producer needs to maintain the output level at $\underline{Q} = 20$ units, what are the values K^* *and* L^* *that minimizes the total cost*, and what is the corresponding minimum cost? Use the bordered Hessian to verify that the second-order sufficient condition is met.
b. Suppose now that the output level is not fixed, but the producer has a budget constraint at \$400. Assume that this producer spends his entire budget on the production. Determine the values K^* *and* L^* *that maximizes the total output.* Also, use the bordered Hessian to verify that the second-order sufficient condition is met.

a) $\min \; 20K + 5L$

$\text{s.t.} \quad 20 = \sqrt{K}\cdot\sqrt{L}$

$$\mathcal{L}(K, L, \lambda; \text{exo}) = 20K + 5L + \lambda(20 - \sqrt{K}\cdot\sqrt{L})$$

$$\frac{\partial\mathcal{L}}{\partial K} = 20 - \lambda\left(\frac{1}{2}K^{-1/2}L^{1/2}\right) = 0 \quad \text{—①}$$

$$\frac{\partial\mathcal{L}}{\partial L} = 5 - \lambda\left(\frac{1}{2}K^{1/2}L^{-1/2}\right) = 0 \quad \text{—②}$$

$$\frac{\partial\mathcal{L}}{\partial \lambda} = 20 - \sqrt{K}\cdot\sqrt{L} = 0 \quad \text{—③}$$

$$\frac{①}{②}; \quad \frac{20}{5} = \frac{-\lambda\left(\frac{1}{2}K^{-1/2}L^{1/2}\right)}{-\lambda\left(\frac{1}{2}K^{1/2}L^{-1/2}\right)}$$

$$4 = \frac{L}{K}$$

$4K = L$ take into ③

$$20 - K^{1/2}\cdot 2(K)^{1/2} = 0$$

$$2K = 20$$

$$K^* = 10$$

then $L^* = 4(10) = 40$

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda K} & L_{\lambda L} \\ L_{K\lambda} & L_{KK} & L_{KL} \\ L_{L\lambda} & L_{LK} & L_{LL} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}K^{-1/2}L^{1/2} & -\frac{1}{2}K^{1/2}L^{-1/2} \\ -\frac{1}{2}K^{-1/2}L^{1/2} & \lambda\frac{1}{4}K^{-3/2}L^{1/2} & -\lambda\frac{1}{4}K^{-1/2}L^{-1/2} \\ -\frac{1}{2}K^{1/2}L^{-1/2} & -\lambda\frac{1}{4}K^{-1/2}L^{-1/2} & \lambda\frac{1}{4}K^{1/2}L^{-1/2} \end{bmatrix}$$

$$|\bar{H}_3| = \left[0 - \frac{1}{16}\lambda K^{0.5}L^{-0.5} - \frac{1}{16}\lambda K^{-0.5}L^{-0.5}\right] - \left[\frac{1}{16}\lambda K^{1.5}L^{-0.5} + \frac{1}{16}\lambda K^{-0.5}L^{0.5}\right] \quad < 0$$

So, $|\bar{H}|$ is negative definite

$|\bar{H}|$ is convex along constraint set

at (K^*, L^*) is equal to $(10, 40)$ the cost is minimized

b. Suppose now that the output level is not fixed, but the producer has a budget constraint at \$400. Assume that this producer spends his entire budget on the production. Determine the values K^* *and* L^* *that maximizes the total output.* Also, use the bordered Hessian to verify that the second-order sufficient condition is met.

From $Q = f(K,L) = K^{0.5}L^{0.5}$

s.t. $20K + 5L = 400$

$$\mathcal{L}(K,L,\lambda,exo) = K^{0.5}L^{0.5} + \lambda(400-20K-5L)$$

$$\frac{\partial \mathcal{L}}{\partial K} = \frac{1}{2}K^{-0.5}L^{0.5} - 20\lambda = 0 \quad \text{— (1)}$$

$$\frac{\partial \mathcal{L}}{\partial L} = \frac{1}{2}K^{0.5}L^{-0.5} - 5\lambda = 0 \quad \text{— (2)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 400 - 20K - 5L = 0 \quad \text{— (3)}$$

$$\frac{(1)}{(2)}; \quad \frac{K^{-0.5}L^{0.5}}{K^{0.5}L^{-0.5}} = 4$$

$$K^{-1}L = 4$$

$$L = 4K$$

From $400-20K-5(4K) = 0$

$$400-20K-20K = 0$$

$$K^* = 10$$

From $400-20(10)-5L = 0$

$$L^* = 40$$

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda K} & L_{\lambda L} \\ L_{K\lambda} & L_{KK} & L_{KL} \\ L_{L\lambda} & L_{LK} & L_{LL} \end{bmatrix} = \begin{bmatrix} 0 & -20 & -5 \\ -20 & -\frac{1}{4}K^{-1.5}L^{0.5} & \frac{1}{4}K^{-0.5}L^{-0.5} \\ -5 & \frac{1}{4}K^{-0.5}L^{-0.5} & -\frac{1}{4}K^{0.5}L^{-1.5} \end{bmatrix}$$

$$|\bar{H}_3| = \left(0 + 25K^{-0.5}L^{-0.5} + 25K^{-0.5}L^{-0.5}\right) - \left(-\frac{25}{4}K^{-1.5}L^{0.5} - 100K^{0.5}L^{-1.5}\right) > 0$$

So, $|\bar{H}|$ is positive definite

$|\bar{H}|$ is concave along constraint set

at (K^*, L^*) is equal to $(10, 40)$ the total output maximized

Question 3:

Consider a two-period consumption model. The consumer's utility is given by:

$$U(C_1, C_2) = \frac{C_1^{1-\theta}}{1-\theta} + \beta \frac{C_2^{1-\theta}}{1-\theta},$$

where C_1 and C_2 are consumption levels in period 1 and period 2, respectively; θ measures the degree of relative risk aversion where $0 < \theta < 1$; and β is the discount factor where $0 < \beta < 1$.

The consumer's budget constraint is given by:

$$C_1 + \frac{C_2}{1+r} = I_1,$$

where r is the interest rate ($0 < r < 1$), and I_1 is the income in period 1.

Answer the following questions.

a) Write down the utility maximization problem subject to the above budget constraint. Use Lagrange method to derive the optimal values of C_1^* and C_2^* in terms of the parameters θ, β, r, and I_1.

b) Based on your answer in (3.1), derive the impact of an increase in the interest rate on the optimal consumption in period 1 ($\frac{\partial C_1^*}{\partial r}$). Interpret its economic meaning.

c) Verify your answer in (a) by using the second-order sufficient condition.

a. $\mathcal{L} = \frac{C_1^{1-\theta}}{1-\theta} + \beta \frac{C_2^{1-\theta}}{1-\theta} + \lambda \left[I_1 - C_1 - \frac{C_2}{1+r} \right]$

F.O.C : $[C_1] = \frac{d\mathcal{L}}{dC_1} = \frac{\cancel{(1-\theta)}^{1}}{\cancel{(1-\theta)}_{1}} \cdot C_1^{-\theta} + (-\lambda) = 0 \;:\; C_1 = \lambda^{-\frac{1}{\theta}}$

$[C_2] = \frac{d\mathcal{L}}{dC_2} = \beta \cdot \frac{\cancel{(1-\theta)}^{1}}{\cancel{(1-\theta)}} \cdot C_2^{-\theta} - \frac{\lambda}{1+r} = 0 \;:\; C_2 = \left(\frac{\lambda}{\beta(1+r)} \right)^{-\frac{1}{\theta}}$

$[\lambda] = \frac{d\mathcal{L}}{d\lambda} = I_1 - C_1 - \frac{C_2}{1+r} = 0 \;:\; I_1 = C_1 + \frac{C_2}{1+r}$

$$I_1 = \frac{1}{\lambda^{\frac{1}{\theta}}} + \frac{\left(\frac{\beta(1+r)}{\lambda} \right)^{\frac{1}{\theta}}}{1+r}$$

$$= \frac{1}{\lambda^{\frac{1}{\theta}}} + \frac{\beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}}}{\lambda^{\frac{1}{\theta}} \cdot (1+r)}$$

$$I_1 = (1+r) + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}}$$

$$\lambda^{\frac{1}{\theta}} \cdot (1+r)$$

$$\lambda = \left[\frac{(1+r) + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}}}{I_1 (1+r)} \right]^{\theta}$$

$$\therefore C_1 = \left[\frac{(1+r) + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}}}{I_1 (1+r)} \right]^{-1} = \frac{I_1 (1+r)}{(1+r) + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}}} \; \#$$

$$C_2 = \left[\left[\frac{(1+r) + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}}}{I_1 (1+r)} \right]^{\theta} \div (1+r) \right]^{-\frac{1}{\theta}} = \frac{I_1 (1+r) \cdot (1+r)^{\frac{1}{\theta}}}{(1+r) + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1}{\theta}}} \; \#$$

b. $\frac{dC_1}{dr} = \frac{I_1}{1 + \beta^{\frac{1}{\theta}} \cdot \frac{1}{\theta} (1+r)^{\frac{1}{\theta}-1} \cdot 1}$; $0 < \theta < 1$ and $0 < \beta < 1$

$\therefore$ when r increase, C_1 will be decreased

c.

$$\bar{H} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda C_1} & \mathcal{L}_{\lambda C_2} \\ \mathcal{L}_{C_1 \lambda} & \mathcal{L}_{C_1 C_1} & \mathcal{L}_{C_1 C_2} \\ \mathcal{L}_{C_2 \lambda} & \mathcal{L}_{C_2 C_1} & \mathcal{L}_{C_2 C_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & \frac{-1}{1+r} \\ -1 & -\theta \cdot C_1^{-1-\theta} & 0 \\ \frac{-1}{1+r} & 0 & -\theta\beta \cdot C_2^{-\theta-1} \end{bmatrix}$$

$$|\bar{H}| = \left[\begin{array}{ccc} 0 & -1 & \frac{-1}{1+r} \\ -1 & -\theta \cdot c_1^{-1-\theta} & 0 \\ \frac{-1}{1+r} & 0 & -\theta\beta \cdot c_2^{-\theta-1} \end{array}\right] \begin{array}{cc} 0 & -1 \\ -1 & -\theta \cdot c_1^{-1-\theta} \\ \frac{-1}{1+r} & 0 \end{array}$$

$$= (0+0+0) - \left(\left(\frac{-1}{1+r} \cdot -\theta c_1^{-1-\theta} \cdot \frac{-1}{1+r} \right) + 0 - \theta\beta \cdot c_2^{-\theta-1} \right)$$

$$= \left(\frac{1}{1+r} \cdot \theta c_1^{-1-\theta} \cdot \frac{1}{1+r} \right) + \theta\beta \cdot c_2^{-\theta-1}$$

$\because$ $0 < \theta < 1$, $0 < \beta < 1$ and $0 < r < 1$ / c_1 and c_2 is positive in (b)

$\therefore$ $|\bar{H}| > 0 \longrightarrow U(c_1, c_2)$ is concave along the constraint set $\longrightarrow$ maximum