EE 415/418 Game Theory

Lecture 13. Static Games of Incomplete Information

- Static Bayesian Games
- Bayesian Nash Equilibrium

Static Bayesian Games

- Up to this point, we assumed that all relevant information about each player, including payoffs, are common knowledge.
- In many circumstances, players have incomplete information, not knowing another player's payoff function.
- For example, competing firms in an industry may not know each other cost, thereby not knowing other firms' payoff function.
- In a seal-bid auction, you know your valuation for the good, but not that for others.
- Payoff then varies with the type of players.

- **Example** – Cournot Competition under Asymmetric Information
- Two firms competes in Cournot fashion.
- The inverse demand function is $p = a - Q$.
- Firm 1's cost function is $C_1(q_1) = cq_1$.
- Firm 2's cost function is $C_1(q_1) = c_H q_1$ with probability θ and $C_1(q_1) = c_L q_1$ with probability $1 - \theta$, where $c_L < c_H$.

Firm 2 could be a new entrant or just has a new technology. All of this is common knowledge.

- If firm 2 is of high cost type, its problem is to solve

$$\max_{q_2}[(a - q_1^* - q_2) - c_H]q_2$$

- If it is of low cost type, it solves

$$\max_{q_2}[(a - q_1^* - q_2) - c_L]q_2$$

- Firms 1's problem is to maximize its expected profit

$$\max_{q_1}\{\theta[(a - q_2^*(c_H) - q_1) - c]q_2 + (1 - \theta)[(a - q_2^*(c_L) - q_1) - c]q_2\}$$

- FOCs are given by

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2},$$

$$q_2^*(c_L) = \frac{a - q_1^* - c_H}{2},$$

$$q_1^* = \frac{\theta[a - q_2^*(c_H) - c] + (1-\theta)[a - q_2^*(c_L) - c]}{2}.$$

- Solving the three equations with three unknowns gives

$$q_2^*(c_H) = \frac{a - 2c_H + c}{2} + \frac{1-\theta}{6}(c_H - c_L),$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{2} \square \frac{\theta}{6}(c_H - c_L),$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}.$$

- For firm 1, its output is less than if knowing firm 2's type.

- For firm 2, high type will produce more than that of complete information (since even high-cost type makes them producing less but knowing that firm 1 cannot do a good response (too little), thus firm 2 can produce more)
- For firm 2 with low type will produce less than that of complete information.

3.1B Normal Form Representation of Bayesian Game

- An n-player static Bayesian Game G consists of the players' action spaces A, type spaces T, their belief p; and their payoff function u.

- We need to consider a player's beliefs about other players' payoff (preference or type), his beliefs about other players' beliefs about his payoff (preference or type), and so on.
- Each player type *i* corresponds to a different payoff.
- Thus, player *i*'s payoff function: $u_i\ (a_1, ..a_n; t_i)$ where t_i is called player i's type from set T_i.
- Example, from Cournot game $T_2 = \{c_H, c_L\}$.
- So, if player *i* knows his payoff, then this is to say that player *i* knows his type.
- Equivalently, if you do not know other player's payoff ==not know the types of other player.

- Let denote player i's belief about the other players' types as $p_i(t_{-i})$ (assuming not dependent with t_i)
- Harsanyi proposed to convert this game to games with imperfect information instead.
- By assuming that Nature moves first, and assigns type or reveals type to particular player, but not others. Then players with each private information play game simultaneously.
- Thus, it looks like players at some moves in the game do not know the complete history thus far.
- The distribution of types (random variables) is assumed to be common knowledge. (prob.about type that Nature will draw)

- Once nature reveals type to such player, he can compute the belief about other players' type as follows:

$$p(t_{-i} \mid t_i) = \frac{p(t_i, t_{-i})}{\sum_{t_{-i} \in T_{-i}} p(t_i, t_{-i})}.$$

- This is Bayes' rule, conditional probability.
- Generally, we will assume that this is reduced to $p(t_{-i})$. Thus, player j knows about i's belief about j type.

- Given timing of this game, Nature begins the game by drawing the players' types, a strategy for player *i* must specify a feasible action for each of player *i*'s possible types.
- A strategy for player *i* is a function of type, $s_i(t_i)$.
- But since other players doesn't know player i's type, thus other players' strategies depend on what player i would do for any given player i's type.
- Thus, player i will have to think about what he would do for all possible types his opponents think he might be. (his action if other types are drawn).
- In Cournot game, player 2's action is thus output if cost is low and if cost is high

Definition: Bayesian Nash Equilibrium

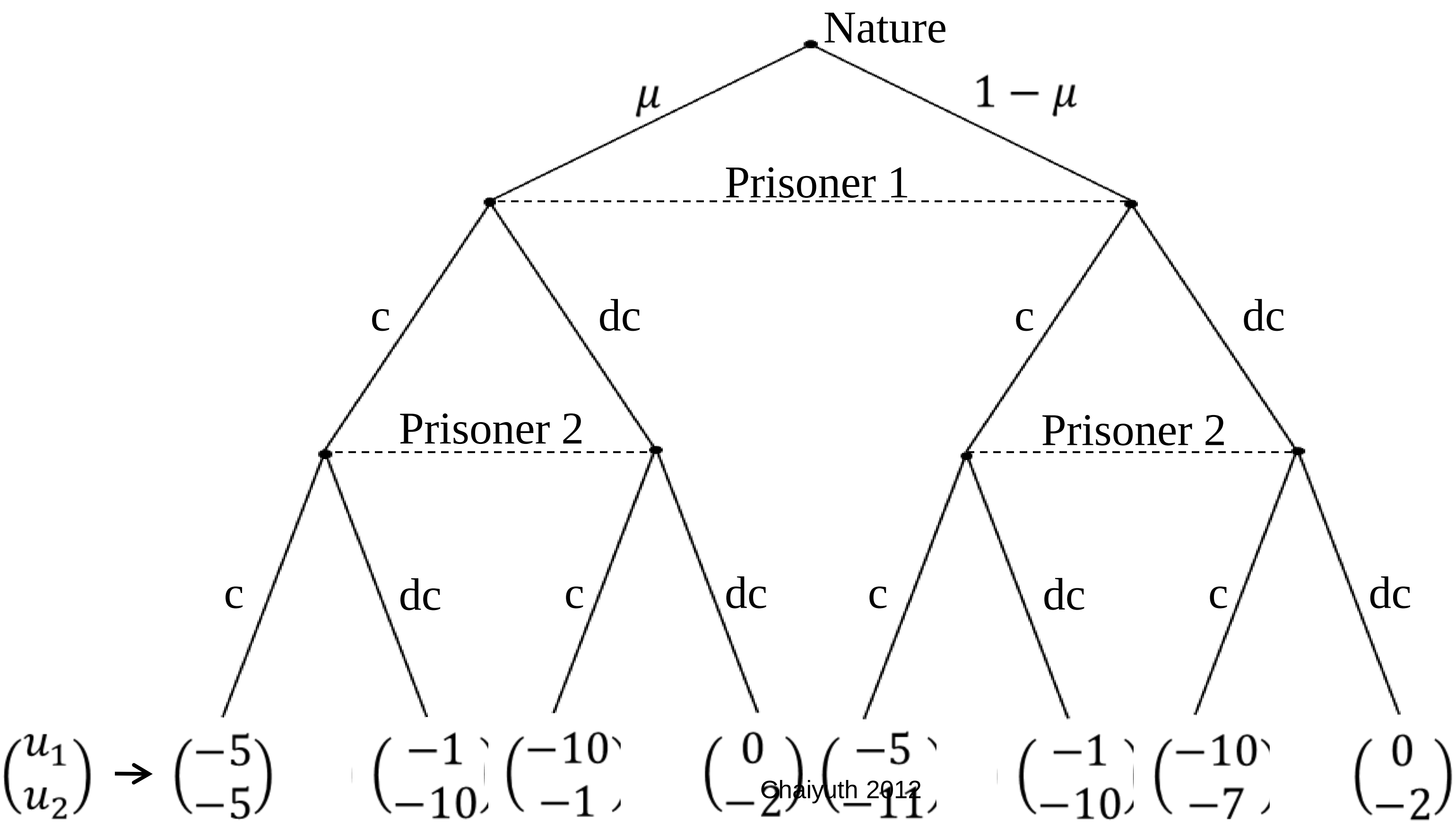
- Strategy profile *s** are BNE if for each player i and for each of i's types,

$$s_i^*(t_i)\, solves \max \sum_{t_{-i}} u_i(a_i, s_{-i}^*(t_{-i}); t).p_i(t_{-i})$$

Ex. – Modification of Prisoners' Dilemma

- There is only one possible type of player 1, called type I, and there are two possible type of prisoner 2. With probability μ, prisoner 2's preference is depicted by the extensive form representation (called type I), and with probability $1-\mu$, prisoner 2 hates to rat on his accomplice (called type II) so that ratting has a psychic penalty equals to 6 years in prison.
- Player 1 has two pure strategies while player 2 has four pure strategies (confess if type I, confess if type II), (confess if type I, don't confess if type II), (don't confess if type I, confess if type II), and

Ex. – Modification of Prisoners' Dilemma



$T_1 = \{\text{I}\}$, $T_2 = \{\text{I, II}\}$,
$p(t_1 = \text{I} \mid t_2) = 1$ for all $t_2 \in T_2$,
$p(t_2 = \text{I}) = \mu$, $p(t_2 = \text{II}) = 1 - \mu$,
$S_1 = \{(\text{c}), (\text{dc})\}$,
$S_2 = \{(\text{c if } t_2 = \text{I, c if } t_2 = \text{II}), (\text{c if } t_2 = \text{I, dc if } t_2 = \text{II}),$
$(\text{dc if } t_2 = \text{I, c if } t_2 = \text{II}), (\text{dc if } t_2 = \text{I, dc if } t_2 = \text{II})\}$,
and u_i is given by payoff matrices

Player 2 (I)

Player 1	c	dc
c	-5,-5	-1,-10
dc	-10,-1	0,-2

Player 2 (II)

Player 1	c	dc
c	-5,-11	-1,-10
dc	-10,-7	0,-2

Ex. – Modification of Prisoners' Dilemma

- Consider the case in which player 2 is of Type I first. In this case, Player 1's best response to player 2 strategies is $R_1(c) = c$, $R_1(dc) = dc$. Player 2's best response function is $R_2(c) = c$, $R_2(dc) = c$. Thus, if player 2 is of type I, c is his dominant strategy.
- If player 2 is of type II, then $R_1(c) = c$, $R_1(dc) = dc$, $R_2(c) = dc$, $R_2(dc) = dc$. Thus, if player 2 is of type II, dc is his dominant strategy.

- Player 1 is unsure about player 2 type, given player 2's best response function, player 1's expected payoff gained by playing c is
$$\mu(-5) + (1-\mu)(-1) = -1 - 4\mu$$
- If player 1 play dc then his expected payoff is
$$\mu(-10) + (1-\mu)(0) = -10\mu$$
- For all $\mu \in [0, \frac{1}{6})$, playing c yields greater expected payoff. For all $\mu \in (\frac{1}{6}, 1]$, playing dc yields greater expected payoff.

- Player 1 will play c if $\mu \in [0,1/6)$, and will play dc if $\mu \in (1/6,1]$.
- BNE are (c, c if type I, dc if type II) if $\mu \in [0,1/6)$, and (dc, dc if type I, dc if type II) if $\mu \in (1/6,1]$.