Suppose that there are three firms. Each firm produces one single product that is imperfectly substitutable to the other two products, i.e. differentiated products. Each of the firm *strategically* competes with each other, and faces the following market demand equations given by,

$$Q_1 = 80 - 2P_1 + P_{23}$$

$$Q_2 = 80 - 2P_2 + P_{13}$$

$$Q_3 = 80 - 2P_3 + P_{12}$$

where P_{23} is the average of the prices charged by firms 2 and 3, P_{13} is the average of the prices charged by firms 2, P_{12} is the average of the prices charged by firms 1 and 2 [e.g., $P_{12} = 0.5(P_1 + P_2)$]. Suppose that each of the three firms, as indexed by j, has the cost function given by $C^j(Q_j) = c_j Q_j$.

Consider the following problems.

a) Set up the profit function of each individual firm, and derive the profit-maximizing condition of each of the three firms when each of them *optimally* sets for the level of price that maximizes its own profit. (**Hint**: *Derive the best response function of each firm, i.e. optimal contingent plan or reaction function.*)

$$\Pi_1 = P_1(80 - 2P_1 + 0.5P_2 + 0.5P_3) - C(80 - 2P_1 + 0.5P_2 + 0.5P_3)$$

$$\frac{\partial \Pi}{\partial P_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 2C_1 = 0 \qquad P_1 = \frac{80 + 0.5P_2 + 0.5P_3 + 2C_1}{4} = 20 + \frac{P_2}{8} + \frac{P_3}{8} + \frac{4C_1}{8}$$

$$\frac{\partial \Pi}{\partial P_2} = 80 - 4P_2 + 0.5P_1 + 0.5P_3 + 2C_2 = 0 \qquad P_2 = \frac{80 + 0.5P_1 + 0.5P_3 + 2}{4} = 20 + \frac{P_1}{8} + \frac{P_3}{8} + \frac{4C_2}{8}$$

$$\frac{\partial \Pi}{\partial P_2} = 80 - 4P_3 + 0.5P_1 + 0.5P_2 + 2C_3 = 0 \qquad P_3 = \frac{80 + 0.5P_1 + 0.5P_2 + 2}{4} = 20 + \frac{P_1}{8} + \frac{P_2}{8} + \frac{4C_3}{8}$$

b) Suppose that Based on (a), what would be the *equilibrium* level of price that each of the firms choose. Calculate the level of profit that each firm yields. (Hint: *Solve for the Bertrand equilibrium with differentiated product*)

$$P_1 = 20 + \frac{P_2}{8} + \frac{P_3}{8} + \frac{4}{8}$$

$$20 + \frac{80 + 0.5P_1 + 0.5P_3 + 2C_2}{32} + \frac{80 + 0.5P_1 + 0.5P_2 + 2C_3}{32} + \frac{2C_1}{4}$$

$$\frac{640 + 80 + 0.5P_1 + 0.5P_3 + 2C_2 + 80 + 0.5P_1 + 0.5P_2 + 2C_3 + 16C_1}{32}$$

$$P_1 \quad \frac{400 + 0.25P_2 + 0.25P_3 + C_2 + C_3 + 8C_1}{15.5}$$

$$P_2 \quad \frac{400 + 0.25P_1 + 0.25P_3 + C_1 + C_3 + 8C_2}{15.5}$$

$$P_3 \quad \frac{400 + 0.25P_1 + 0.25P_2 + C_1 + C_2 + 8C_3}{15.5}$$

c) How does the optimal price vary with respect to the size of marginal cost?

$$\frac{\Delta P_1, P_2, P_3}{\Delta C} = \frac{8}{15.5}$$ \$ (MC increase per 1 \$)

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where P_{23} is the average of the prices charged by firms 2 and 3, P_{13} is the average of the prices charged by firms 2, P_{12} is the average of the prices charged by firms 1 and 2 [e.g., $P_{12} = 0.5(P_1 + P_2)$]. Suppose that each of the three firms, as indexed by j, has the cost function given by $C^j(Q_j) = c_j Q_j$.

Now suppose that all the three firms agree to operate under a cartel (collusive) agreement. That is, they consider a joint pricing scheme that maximizes the joint profit function of the three firms combined. Consider the following problems.

d) Construct the joint profit function.

$$\Pi = TR - TC$$

$$\Pi = P_1Q_1 + P_2Q_2 + P_3Q_3 - C_1Q_1 - C_2Q_2 - C_3Q_3$$

$$= 80P_1 - 2P_1^2 - 0.5P_1P_2 + 0.5P_1P_3) + (80P_2 - 2P_2^2 + 0.5P_1P_2 + 0.5P_2P_3) + (80P_3 - 2P_3^2 + 0.5P_1P_3 + 0.5P_2P_3)$$

$$-(80C_1 - 2P_1C_1 + 0.5P_2C_1 + 0.5P_3C_1) - (80C_2 - 2P_2C_2 + 0.5P_1C_2 + 0.5P_3C_2) - (80C_3 - 2P_3C_3 + 0.5P_1C_3 + 0.5P_2C_3)$$

e) Given that $c_j = c,\ \forall j$. Calculate the level of optimal price setting that each of the firm will set under the joint profit function problem. Confirm your result with the second-order derivative test.

$$\frac{d\Pi}{dP_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 0.5P_2 + 0.5P_3 + 2C_1 - 0.5C_2 - 0.5C_3$$
$$= 80 - 4P_1 + P_2 + P_3 + 2C_1 - 0.5C_2 - 0.5C_3$$

$$\frac{d\Pi}{dP_2} = 0.5P_1 + 80 - 4P_2 + 0.5P_1 + 0.5P_3 + 0.5P_3 - 0.5C_1 + 2C_2 - 0.5C_3$$
$$= 80 - 4P_2 + P_1 + P_3 - 0.5C_1 + 2C_2 - 0.5C_3$$

$$\frac{d\Pi}{dP_3} = 0.5P_1 + 0.5P_2 + 80 - 4P_3 + 0.5P_1 + 0.5P_2 - 0.5C_1 - 0.5C_2 + 2C_3$$
$$= 80 - 4P_3 + P_1 + P_2 - 0.5C_1 - 0.5C_2 + 2C_3$$

$$80 - 4P_1 + P_2 + P_3 + 2C_1 - 0.5C_2 - 0.5C_3 \quad —(1)$$
$$80 - 4P_2 + P_1 + P_3 - 0.5C_1 + 2C_2 - 0.5C_3 \quad —(2)$$
$$80 - 4P_3 + P_1 + P_2 - 0.5C_1 - 0.5C_2 + 2C_3 \quad —(3)$$

$(1) = (2)$; $4P_1 - P_2 - P_3 = -P_1 + 4P_2 - P_3$

$P_1 = P_2$

plug $P_1 = P_2$ into (1) and (3)

$4P_1 - P_1 - P_3 = -P_1 - P_1 + 4P_3$

$P_1 = P_3$

plug $P_1 = P_2 = P_3$ into (1)

$4P_1 - P_1 - P_1 = 80 - C$

$P_1 = P_2 = P_3 = \frac{80 - C}{3}$ #

SOC.

$$H = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{bmatrix}$$

$|H_1| < 0$

$|H_2| = 15 > 0$

$|H_3| = -50 < 0$

H is negative definite $\to$ $d^2\Pi < 0$, globally concave

f) Is the Cartel agreement self-sustaining? Why?

Cartel agreement is not self-sustaining even though all 3 producers of the firms are different. The quantity demanded of each good is interdependent on one another. If one of the firm were to set price a little bit higher than it is in the agreement, the quantity demanded for other products will drop. Therefore, the cheated firm will gain relatively higher profit compared to the rest.

g) Revisit question (e) and assume that the marginal costs are varied with respect to firms. What would be the implication of the collusive equilibrium?

The implication of collusive would be that if MC are varied wrt. firms, they should set the price of each product based on MC of all 3 goods. That is, if the price of product 1 has higher MC, holding all other things constant, the firm should sell it at relatively.
At the end, they should satisfy the condition where $MR(Q_1) = MR(Q_2) = MR(Q_3) = MC(Q_{total})$

Question 2:

Pakorn's utility function depends on the consumption of two commodities, x and y, and it is given by

$$U(x, y) = 2xy$$

Suppose that his income is \$72, and the prices per unit of x and y are \$4 and \$6, respectively. Assume that Pakorn spends all of his income, and the values of x and y are both non-zero.

a. Use the Lagrange method to determine the values of x^* and y^* that maximize Pakorn's utility given an income constraint. Verify that the second-order sufficient conditions are satisfied.
b. Determine the maximum utility level and the Lagrange multiplier. Interpret the economic interpretation of the Lagrange multiplier.
c. Suppose that the income is now \$73. Approximate the new maximum utility level.

$4x + 6y = 72 \qquad x, y > 0$

a. $\mathcal{L}(x, y, \lambda; a_0, a_1, c) = U(x, y) + \lambda[c - g(x, y)]$

$$= 2xy + \lambda[72 - 4x - 6y]$$

F.O.C

$[x]: \dfrac{\partial f}{\partial x} = 2y - 4\lambda = 0 \rightarrow y = 2\lambda$ — (1)

$[y]: \dfrac{\partial f}{\partial y} = 2x - 6\lambda = 0 \rightarrow x = 3\lambda$ — (2)

$[\lambda]: \dfrac{\partial f}{\partial \lambda} = 72 - 4x - 6y = 0 \rightarrow 4x + 6y = 72$ — (3)

Substitute (1), (2) into (3) to find λ

$4(3\lambda) + 6(2\lambda) = 72$

$12\lambda + 12\lambda = 72$

$\lambda^* = 3$

$y = 2(3) = 6$

$x = 3(3) = 9$

$\therefore x^*, y^*$ that maximizes Pakorn's utility is $(9, 6)$

Check S.O.C.

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x} & L_{\lambda y} \\ L_{x\lambda} & L_{xx} & L_{xy} \\ L_{y\lambda} & L_{yx} & L_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -4 & -6 \\ -4 & 0 & 2 \\ -6 & 2 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 0 & -4 & -6 & 0 & -4 \\ -4 & 0 & 2 & -4 & 0 \\ -6 & 2 & 0 & -6 & 2 \end{array}\right] = (0+48+48)-(0+0+0)$$

$$= 96$$

$\therefore \bar{H}$ is positive, f is concave along the constraint set and solution from F.o.c is represent max.

B. $x^* = 9,\ y^* = 6 \rightarrow x^*$ and y^* that maximizes utility

max. utility $= 2xy = 2(9)(6) = 108$ units

Lagrange multiplier

$\dfrac{df^*}{dc} = \lambda^*$ from a. we get $\lambda^* = 3$, it shows per units changes of f^*

when $\lambda^* > 0$, c increases by 1 unit, f^* will increase by λ^* units

C. From Lagrange multiplier $\rightarrow$ we get λ^* equal to 3

old $c = \$72$, $\boxed{\text{new } c = \$73}$ ↑ 1 \$

- when c increase 1 unit, f^* will increase by 3 units

So, new approximation maximum utility level will be $108 + 3 = 111$ units

Question 4:

A consumer has the utility function $U = \ln C + \ln(24 - N)$, where C is consumption and N is labor supply. Her budget constraint is $pC = \bar{M} + wN$, where p is the price of the consumption good, w the wage rate, and $\bar{M}$ the consumer's non-wage income.

(a) Formulate the problem of utility maximization subject to the budget constraint, and derive the first-order conditions, using the Lagrange multiplier approach and ignoring the nonnegativity constraints.

$$\mathcal{L}(C, N, \lambda; p, \bar{m}, w) = \ln C + \ln(24 - N) + \lambda[\bar{m} + wN - pC]$$

F.O.C.

[C] $\frac{\partial \mathcal{L}}{\partial C} = \frac{1}{C} - \lambda p = 0$ $\quad C\lambda p = 1 \quad C = \frac{1}{\lambda p}$ (1)

[N] $\frac{\partial \mathcal{L}}{\partial N} = w\lambda + \frac{-1}{24-N} = 0$ $\quad 24w\lambda - Nw\lambda - 1 = 0$

$Nw\lambda = 24w\lambda - 1$

$N = 24 - \frac{1}{w\lambda}$

[λ] $\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{m} + wN - pC = 0$

(2) (1)

$\bar{m} + w\left(24 - \frac{1}{w\lambda}\right) - p\left(\frac{1}{\lambda p}\right) = 0$

$\bar{m} + 24w - \frac{1}{\lambda} - \frac{1}{\lambda} = 0$

$\bar{m} + 24w = \frac{2}{\lambda}$

$\lambda\bar{m} + \lambda 24 w = 2$

$\lambda^* = \frac{2}{\bar{m} + 24w}$ (3)

(1) $C^* = \frac{\bar{m} + 24w}{2p}$

(2) $N^* = 24 - \frac{\bar{m} - 24w}{2w}$

(b) Find the demand function $C = C^*(p, w, \bar{M})$ and the labor supply function $N = N^*(p, w, \bar{M})$ (i.e., express C and N in terms of p, w, and $\bar{M}$). Show that N^* and C^* are homogeneous of degree zero in $(p, w, \bar{M})$.

$C^* = \frac{\bar{m} + 24w}{2p}$

$\frac{t\bar{m} + 24wt}{2tp}$

$= \frac{\cancel{t}(\bar{m} + 24w)}{2\cancel{t}p}$

$= \frac{\bar{m} + 24w}{2p}$ $\quad t^0$ HM degree $= 0$

$N^* = 24 - \frac{\bar{m} - 24w}{2w}$

$= \frac{48w - \bar{m} - 24w}{2w}$

$\frac{48tw - t\bar{m} - 24tw}{2tw} = \frac{\cancel{t}(48w - \bar{m} - 24w)}{\cancel{t}2w}$

$= \frac{48w - \bar{m} - 24w}{2w}$

t^0 HM degree $= 0$

(c) Let $U^* = \ln[C^*(p, w, \bar{M})] + \ln[24 - N^*(p, w, \bar{M})]$. Show that $\partial U^*/\partial \bar{M} > 0$ and $\partial U^*/\partial p < 0$. Show that U^* is concave in $\bar{M}$ and convex in p. What is the relationship between $\partial U^*/\partial \bar{M}$ and the Lagrange multiplier?

$$U^* = \ln\left(\frac{\bar{M} + 24w}{2p}\right) + \ln\left[\cancel{24} - \cancel{24} + \frac{\bar{M} + 24w}{2w}\right]$$

$$p^{-1}\left(\frac{\bar{M}+24w}{2}\right)$$

$$= \ln\left(\frac{\bar{M} + 24w}{2p}\right) + \ln\left[\frac{\bar{M} + 24w}{2w}\right]$$

$$\frac{\partial U^*}{\partial \bar{M}} = \frac{\cancel{2p}}{\bar{M}+24w}\left(\frac{1}{\cancel{2p}}\right) + \frac{\cancel{2w}}{\bar{M}+24w}\left(\frac{1}{\cancel{2w}}\right)$$

$$= \frac{1}{\bar{M}+24w} + \frac{1}{\bar{M}+24w} = \frac{2}{\bar{M}+24w} > 0$$

$$\frac{\partial U^*}{\partial p} = \frac{2p}{\bar{M}+24w}\left(-\frac{(\bar{M}+24w)}{2p^2}\right) = \frac{\cancel{2p}}{\bar{M}+24w}\left(\frac{-\bar{M}-24w}{\cancel{2}p\cancel{x}}\right) = \frac{-\bar{M}-24w}{\bar{M}+24w} < 0$$

$$\frac{\partial U^*}{\partial \bar{M}} = \frac{2}{\bar{M}+24w} = \lambda^*$$