Relations & Functions I

1 Introduction

There are many kinds of relationships in the world. For instance, we say that two people are related by blood if they share a common ancestor and that they are related by marriage if one shares a common ancestor with the spouse of the other. We also speak of the relationship between student and teacher, between people who work for the same employer, and between people who share a common ethnic background.

The objects of mathematics may be related in various ways. E.g.

- A set A may be said to be related to a set B if A is a subset of B, or if A is not a subset of B, or if A and B have at least one element in common.

-A number x may be said to be related to a number y if $x < y$, or if x is a factor of y, or if $x^2 + y^2 = 1$.

Definition 1.1 (Ordered Pair)**.** Let A be a given set.
An **ordered pair** (a, b) of elements in A is defined to be the set $\{a, \{a, b\}\}$.

- a is called the *first component* or the *first element.*
- b is called the *second component* or the *second element.*

Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } b = d.$$

Example 1.1. .

- Is $(3, 1) = (1, 3)$?
- Is $(3, \frac{5}{10}) = (\sqrt{9}, \frac{1}{2})$?

Definition 1.2 (Cartesian Product)**.** Let A and B be two given sets.
The **Cartesian product** of A and B, denoted $A \times B$ (and read "A cross B")
is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

Example 1.2. 1. Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

(a) Find $B \times B$.

(b) Find $A \times B$.

(c) Find $B \times A$.

(d) Is $A \times B = B \times A$?

(e) How many elements are in $A \times B$, $B \times A$, and$B \times B$?

2. Let $\mathbb{R}$ denote the set of all real numbers.
Describe $\mathbb{R} \times \mathbb{R}$ (this is also called **Cartesian plane**).

2 Relation

There are many kinds of relationships in the world, such as, relationship between people who work for the same company, between parents and children, and between people who live in the same country.

In mathematics, the objects also can be related in various ways, e.g.:

- A set A may be said to be related to a set B if A is a subset of B, or if A is not a subset of B.
- A number x may be said to be "related" to a number y
  - if $x < y$,
  - if x is a factor of y, or
  - if $x^2 + y^2 = 1$.

Definition 2.1. Let A and B be sets. A relation R from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written

$$x \ R \ y,$$

if, and only if, (x, y) is in R.
The set A is called the **domain** of R and the set B is called its **co-domain**.

- The notation for a relation R may be written symbolically as follows:

$$x \ R \ y \ \text{ means that } \ (x, y) \in R.$$

- The notation $x \not R \ y$ means that x is not related to y by R:

$$x \not R \ y \ \text{ means that } \ (x, y) \notin R.$$

Example 2.1. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows: Given any $(x, y) \in A \times B$,

$$(x, y) \in R \ \text{ means that } \ \frac{x - y}{2} \ \text{ is an integer.}$$

1. State explicitly which ordered pairs are in $A \times B$ and which are in R.

2. Is 1 R 3?
   Is 2 R 3 ?
   Is 2 R 2?

3. What are the domain and co-domain of R?

Example 2.2. Define a relation C from $\mathbb{R}$ to $\mathbb{R}$ as follows: For any $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$(x, y) \in C \text{ means that } \; x^2 + y^2 = 1.$$

1. Is $(1, 0) \in C$?

   Is $(0, 0) \in C$?

   Is $(-1/2, \sqrt{3}/2) \in C$?

   Is $-2 \; C \; 0$?

   Is $0 \; C \; (-1)$?

   Is $1 \; C \; 1$?

2. What are the domain and co-domain of C?

3. Draw a graph for C by plotting the points of C in the Cartesian plane.

2.1 Arrow Diagram of a Relation

Arrow diagram of a relation

Suppose R is a relation from a set A to a set B. The arrow diagram for R is obtained as follows:

1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R. Symbolically:

   Draw an arrow from x to y if, and only if, $x\ R\ y$ or $(x, y) \in R$.

Example 2.3. Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows: For all $(x, y) \in A \times B$,

(i)

$$(x, y) \in S \text{ means that } x < y$$

(ii)

$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for S and T.

3 Functions

A function is a special case of a relation. A **function** f from a set X to a set Y is a relation from X to Y such that for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$. A formal definition of function is given below.

Definition 3.1 (Function)**.** A function f from a set X to a set Y, denoted $f : X \to Y$, is a relation from X to Y that satisfies the following two properties.

(1) Every element in X is related to some element in Y. I.e.

$$\boxed{\forall x \in X,\ \exists y \in Y, \text{ such that } (x, y) \in f.}$$

For every element x in X, there is an element y in Y such that $(x, y) \in f$.
⇒ Every element of A is the first element(component) of an ordered pair of f.

(2) No element in X is related to more than one element in Y. I.e.

$$\boxed{\forall x \in X, \text{ if } (x, y_1), (x, y_2) \in f \text{ where } y_1, y_2 \in Y, \text{ then } y_1 = y_2}$$

For all elements x in X and y_1 and y_2 in Y, if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.
⇒ No two distinct ordered pairs in f have the same first element(component).

The set X is called the **domain** of f and Y is called the **co-domain**of f.

- The set of all values of f taken together is called the **range** of f or the **image** of X under f. I.e.,

$$\text{range of } f = \text{image of } X \text{ under } f = \{y \in Y | y = f(x),\ \text{forsome} x \in X\}.$$

- Given an element y in Y , there may exist elements in X with y as their image. If $f(x) = y$, then x is called a **preimage** of y or an **inverse image** of y. The set of all inverse images of y is called the **inverse image** of y. I.e.,

$$\text{the inverse image of } y = \{x \in X | f(x) = y\}$$

- For $(x, y) \in f$ we use the notation $y = f(x)$. We call y the **image** of x under f.
- E.g. Let $f(x) = x^2$ be a function with domain $\mathbb{R}$. Then

  the inverse image of 4 under $f = \{x \in \mathbb{R} : f(x) = 4\} =$

- A "function" is not **well defined** if it fails to satisfy at least one of the requirements for being a function.

Example 3.1. 1. Show that the relation $f = \{(1, a), (2, b), (3, a)\}$ defines a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$. Find its range.

2. Show that the relation $f = \{(1, a), (2, b), (3, c), (1, b)\}$ does not define a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$.

Example 3.2. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R and S defined below are functions from A to B ?

1. $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$

2. For all $(x, y) \in A \times B, (x, y) \in S$ means that $y = x + 1$.

Let f be a function from a set X to a set Y, denoted $f : X \to Y$. Let $(x, y) \in f$,

- We say that "f sends x to y" or "f maps x to y" and
- write $f : x \to y$ or $f(x) = y$.
- The unique element to which f sends x is denoted $f(x)$ and is called
  - f of x, or
  - the output of f for the input x, or
  - the value of f at x, or
  - the image of x under f.

Diagrams of Functions Recall that if X and Y are finite sets, you can define a function f from X to Y by drawing an arrow diagram. You make a list of elements in X and a list of elements in Y, and draw an arrow from each element in X to the corresponding element in Y. The arrow diagram that defines a function must have the following two properties.

1. Every element of X has an arrow coming out of it.
2. No element of X has two arrows coming out of it that point to two different elements of Y.

Equality of Functions
If $F : X \to Y$ and $G : X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Definition 3.2 (Graph)**.** Let A and B be subsets of $\mathbb{R}$. A function $f : A \to B$ is called a real-valued function of a real variable. In this case, each ordered pair $(x, f(x))$ can be represented by a point in the Cartesian plane. The collection of all such points is called the **graph of** f.

Definition 3.3 (Increasing/Decreasing Function)**.** Let X be the domain of a function f and $S \subset X$.

- f is increasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- f is decreasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Example 3.3. Graph the functions $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$ on the closed interval $[-4, 4]$.

Example 3.4. Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 3$ is increasing on $\mathbb{R}$.

3.1 Examples of Functions

- **The Identity Function on a Set:** Given a set X, define a function I_X from X to X by

$$I_X(x) = x$$

for all x in X.

- **Sequences:** The formal definition of sequences specifies that an infinite sequence is a function defined on the *set of integers that are greater than or equal to a particular integer.* E.g. the sequence denoted

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, \frac{(-1)^n}{n+1}, \ldots.$$

can be thought of as the function f from the nonnegative integers to the real numbers that associates

$$0 \to 1, 1 \to -\frac{1}{2}, 2 \to \frac{1}{3}, 3 \to -\frac{1}{4}, 4 \to \frac{1}{5}, \ldots, n \to \frac{(-1)^n}{n+1}, \ldots.$$

That is, $f : \mathbb{Z}^+ \cup \{0\} \to \mathbb{R}$ is the function defined as follows:

$$f(n) = \frac{(-1)^n}{n+1}.$$

Define a function g from the positive integers to the real numbers for the above sequence, $g : \mathbb{Z}^+ \to \mathbb{R}$,

$$g(x) =$$

- **A Function Defined on a Power Set:** $\mathcal{P}(A)$ denotes the set of all subsets of the set A. Define a function $F : \mathcal{P}(\{a, b, c\}) \to \mathbb{Z}^+ \cup \{0\}$ as follows. For each $X \in \mathcal{P}(\{a, b, c\})$,

$$F(X) = \text{the number of elements in } X.$$

Draw an arrow diagram for F.

- **Functions Defined on a Cartesian Product:** E.g., define functions $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as:
  for all ordered pairs (a, b) of real numbers, $M(a, b) = ab$ and $R(a, b) = (-a, b)$. Find the following:
  $M(-1, -1) =$

  $R(-1, -1) =$

  $M(-\sqrt{2}, \sqrt{2}) =$

  Exercise: Explain why each of M and R is indeed a function.

3.2 Types of Functions

We will consider the following types of functions.

- Injective (one-to-one) functions
- Surjective (onto) functions
- Bijective (one-to-one and onto) functions

3.2.1 Injective (one-to-one) functions

Definition 3.4 (Injective or one-to-one functions)**.** Let F be a function from a set X to a set Y. F is **one-to-one** or **injective** if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently (by contrapositive),

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

A function $F : X \to Y$ is **not** one-to-one if, and only if,

$$\exists x_1, x_2 \in X \text{ with } F(x_1) = F(x_2) \text{ and } x_1 \neq x_2.$$

In terms of arrow diagrams,

- A one-to-one function takes **distinct** points of the domain to **distinct** points of the co-domain.
- A function is not one-to-one will have at least two points of the domain taken to the same point of the co-domain.

Example 3.5. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H : X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$.
Define $K : X \to Y$ as follows: $K(1) = d, K(2) = b$, and $K(3) = d$.
Determine if each of the functions H and K is one-to-one.

One-to-One Functions on Infinite Sets
Let f be a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

- To prove f is one-to-one, we will generally use the method of *direct proof*:

  suppose x_1 and x_2 are elements of X such that
  $f(x_1) = f(x_2)$ and show that $x_1 = x_2$.

- To show that f is **not** one-to-one, we will generally find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Example 3.6. Show that the identity function I_X on a set X is injective.

Example 3.7. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{Z}$ by the rules
$f(x) = 4x - 1$ for all $x \in \mathbb{R}$, and $g(n) = n^2$ for all $n \in \mathbb{Z}$.

1. Is f one-to-one? Prove or give a counterexample.

2. Is g one-to-one? Prove or give a counterexample.

Example 3.8. Show that if $f : \mathbb{R} \to \mathbb{R}$ is increasing then f is one-to-one.

3.2.2 Surjective (onto) functions

Definition 3.5 (Onto or Surjective Functions)**.** Let F be a function from a set X to a set Y. F is onto (or surjective) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$. $F : X \to Y$ is onto if, and only if

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

A function is not onto if, and only if,

$$\exists y \in Y \text{ such that } \forall x \in X, F(x) \neq y,$$

i.e., there is some element in Y that is not the image of any element in X.

Example 3.9. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$.
Define $H : X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$.
Define $K : X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Determine whether each of the functions H and K are surjective (onto) or not.

Example 3.10. (a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is surjective.

(b) Show that the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = 3n - 5$ is not surjective.

(c) Let A and B be two nonempty sets. The functions $P_A : A \times B \to A$ defined by $P_A(a, b) = a$ is called projection function. Show that P_A is surjective.

3.2.3 Bijective (one-to-one correspondence) functions

Definition 3.6 (**one-to-one correspondence** or **bijection**). A **one-to-one correspondence** or **bijection** from a set X to a set Y is a function $F : X \to Y$ that is both *one-to-one* and *onto*.

Example 3.11.

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is a bijective function.

2. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not bijective.

3. (Exercise) Show that the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = x^2$ is bijective.

Example 3.12. (A Function of Two Variables): Define a function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as :

$$\text{For all } (x, y) \in \mathbb{R} \times \mathbb{R},\ F(x, y) = (x + y, x - y).$$

Is F a one-to-one correspondence from $\mathbb{R} \times \mathbb{R}$ to itself?