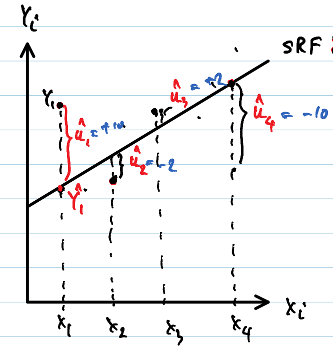
Wednesday, August 29, 2012
12:12 PM

TASK: WE AIM AT ESTIMATING β_1 AND β_2 FROM SRF.



SRF: $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$

$Y_i = \hat{Y}_i + \hat{u}_i$

OBJECTIVE: WE WANT THE SRF LINE TO BE AS CLOSE TO Y_i AS POSSIBLE.

OBJECTIVE: $\sum \hat{u}_i = \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4 + \ldots = 0$ X

OBJECTIVE: $\sum_{i=1}^{n} \hat{u}_i^2$ MUST BE MINIMIZED.

$$\text{MIN} \sum_{i=1}^{n} \hat{u}_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2$$

Y_i: ACTUAL Y_i

$\hat{Y}_i$: ESTIMATED Y_i OR PREDICTED Y_i OR FITTED Y_i

$$= \sum_{i=1}^{n} (Y_i - (\hat{\beta}_1 + \hat{\beta}_2 X_i))^2$$

$$\text{MIN} \left(\sum_{i=1}^{n} \hat{u}_i^2 \right) = \sum_{i=1}^{n} (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2$$

SUM OF RESIDUALS SQUARED

OR $\boxed{\min_{\hat{\beta}_1, \hat{\beta}_2} \sum_{i=1}^{n} \hat{u}_i^2 = f(\hat{\beta}_1, \hat{\beta}_2)}$ ⇒ OBJECTIVE FUNCTION

WE USE A METHOD CALLED "LAGRANGIAN METHOD"

$$\frac{\partial \sum_{i=1}^{n} \hat{u}_i^2}{\partial \hat{\beta}_1} = 0.$$

$$\frac{\partial \sum_{i=1}^{n} \hat{u}_i^2}{\partial \hat{\beta}_2} = 0.$$

OBJECTIVE FUNCTION: $\sum_{i=1}^{n} \hat{u}_i^2 = \sum (Y_i - \underbrace{\hat{\beta}_1 - \hat{\beta}_2 X_i}_{= \hat{Y}_i})^2$

$$\frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_1} = 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)(-1) = 0$$

$\equiv$

$$\sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i) = 0 \quad \text{— (1)}$$

$$\frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_2} = 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)(-X_i) = 0 \quad \text{— (2)}$$

w.r.t

FROM (1) : $\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} \hat{\beta}_1 - \sum_{i=1}^{n} \hat{\beta}_2 X_i = 0$ — (3)

$$\sum_{i=1}^{n} Y_i - n \cdot \hat{\beta}_1 - \hat{\beta}_2 \sum_{i=1}^{n} X_i = 0$$

$$\sum_{i=1}^{n} Y_i = n \cdot \hat{\beta}_1 + \hat{\beta}_2 \sum_{i=1}^{n} X_i$$

OR $n\hat{\beta}_1 = \sum_{i=1}^{n} Y_i - \hat{\beta}_2 \sum_{i=1}^{n} X_i$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} Y_i - \hat{\beta}_2 \sum_{i=1}^{n} X_i}{n}$$

$$= \frac{\sum_{i=1}^{n} Y_i}{n} - \hat{\beta}_2 \underbrace{\frac{\sum_{i=1}^{n} X_i}{n}}_{= \bar{X}}$$

$$\boxed{\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}}$$

FROM (2) $2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)(- X_i) = 0$

$$-2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)(X_i) = 0$$

THIS REDUCES TO $\sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i) X_i = 0$

$$\sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} \hat{\beta}_1 X_i - \sum_{i=1}^{n} \hat{\beta}_2 X_i^2 = 0$$

$$\sum_{i=1}^{n} \hat{\beta}_2 X_i^2 = \sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} \hat{\beta}_1 X_i$$

$$\sum_{i=1}^{n} \hat{\beta}_2 X_i^2 = \sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} (\bar{Y} - \hat{\beta}_2 \bar{X}) X_i$$

$$\sum_{i=1}^{n} \hat{\beta}_2 X_i^2 = \sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \left(\frac{\sum_{i=1}^{n} Y_i}{n} - \hat{\beta}_2 \frac{\sum_{i=1}^{n} X_i}{n} \right)$$

$$\sum_{i=1}^{n} \hat{\beta}_2 X_i^2 = \sum_{i=1}^{n} X_i Y_i - \frac{\sum_{i=1}^{n} X_i \sum Y_i}{n} + \frac{\hat{\beta}_2 \left(\sum_{i=1}^{n} X_i \right)^2}{n}$$

$$\sum_{i=1}^{n} \hat{\beta}_2 X_i^2 - \frac{\hat{\beta}_2 \left(\sum_{i=1}^{n} X_i \right)^2}{n} = \sum_{i=1}^{n} X_i Y_i - \frac{\sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{n}$$

$$\frac{\sum_{i=1}^{n} n \hat{\beta}_2 X_i^2 - \hat{\beta}_2 \left(\sum_{i=1}^{n} X_i \right)^2}{\cancel{n}} = \frac{n \sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{\cancel{n}}$$

$$\hat{\beta}_2 \left[n \sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i \right)^2 \right] = n \sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i$$

$$\boxed{\hat{\beta}_2 = \frac{n\sum_{i=1}^{n} X_iY_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}}$$

WHAT'S NEXT?

SHOW THAT $\hat{\beta}_2 = \dfrac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$ WHERE $x_i = X_i - \bar{X}$

$y_i = Y_i - \bar{Y}$

(DEVIATION FORM)

$$\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

$$= \frac{\sum_{i=1}^{n}\left[X_iY_i - \bar{X}Y_i - X_i\bar{Y} + \bar{X}\bar{Y}\right]}{\sum_{i=1}^{n}\left[X_i^2 - 2X_i\bar{X} + \bar{X}^2\right]}$$

$$= \frac{\sum_{i=1}^{n} X_iY_i - \sum_{i=1}^{n}\bar{X}Y_i - \sum_{i=1}^{n} X_i\bar{Y} + \sum_{i=1}^{n}\bar{X}\bar{Y}}{\sum_{i=1}^{n} X_i^2 - 2\sum_{i=1}^{n} X_i\bar{X} + \sum_{i=1}^{n}\bar{X}^2}$$

$$= \frac{\sum_{i=1}^{n} X_iY_i - \frac{\sum_{i=1}^{n} X_i}{n}\sum_{i=1}^{n} Y_i - \frac{\sum_{i=1}^{n} Y_i}{n}\sum_{i=1}^{n} X_i + \cancel{n}\,\frac{\sum_{i=1}^{n} X_i}{\cancel{n}}\,\frac{\sum_{i=1}^{n} Y_i}{n}}{\sum_{i=1}^{n} X_i^2 - 2\frac{\sum_{i=1}^{n} X_i}{n}\sum_{i=1}^{n} X_i + n\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)^2}$$

$$= \frac{\sum_{i=1}^{n} X_iY_i - \frac{2}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i + \frac{\sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{n}}{\sum_{i=1}^{n} X_i^2 - \frac{2}{n}\left(\sum_{i=1}^{n} X_i\right)^2 + \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2}$$

$$= \frac{\sum_{i=1}^{n} X_iY_i - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{\ldots} \times \left(\frac{n}{\ldots}\right)$$

$$= \frac{\sum_{i=1} X_i Y_i - \frac{1}{n} \sum_{i=1} X_i \sum_{i=1} Y_i}{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2} \times \left(\frac{n}{n}\right)$$

$$= \frac{n \sum_{i=1} X_i Y_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{n \sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2} = \hat{\beta}_2$$

END OF THE PROOF.

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i (Y_i - \bar{Y})}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i Y_i - \overset{=0}{\cancel{\sum_{i=1}^{n} x_i}}\bar{Y}}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}.$$

SINCE
$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (X_i - \bar{X}) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \bar{X} = \sum_{i=1}^{n} X_i - n\bar{X}$$
$$= \sum_{i=1}^{n} X_i - \cancel{n} \frac{\sum_{i=1}^{n} X_i}{\cancel{n}}$$
$$= \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} X_i$$
$$= 0.$$

DIY: $\hat{\beta}_2 = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}$.

SHOW THAT $\hat{\beta}_2 = \frac{\sum_i y_i X_i}{\sum_{i=1}^{n} x_i^2}$.

ORDINARY LEAST SQUARE

THE MODEL: $Y_i = \beta_1 + \beta_2 + u_i \overset{OLS}{\Longrightarrow} \hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}.$$

OBSERVATION

SUPPOSE $k_i = \frac{x_i}{\sum_{i=1}^{n} x_i^2}$

THEN $\hat{\beta}_2 = \sum_{i=1}^{n} k_i y_i$.

THEN $\hat{\beta}_2 = \sum_{i=1} k_i y_i$.

(1) $$\sum_{i=1}^{n} k_i = \frac{\sum_{i=1}^{n} (X_i - \bar{X})}{\sum_{i=1}^{n} x_i^2}$$

$$= \frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \bar{X}}{\sum_{i=1}^{n} x_i^2}$$

$$= \frac{\sum_{i=1}^{n} X_i - n\bar{X}}{\sum_{i=1}^{n} x_i^2}$$

$$= \frac{\sum_{i=1}^{n} X_i - \cancel{n}\dfrac{\sum_{i=1}^{n} X_i}{\cancel{n}}}{\sum_{i=1}^{n} x_i^2}$$

$$\boxed{\sum_{i=1}^{n} k_i = 0}.$$

(2) $$\sum_{i=1}^{n} k_i^2 = \sum_{i=1}^{n} \left(\frac{x_i}{\sum_{i=1}^{n} x_i^2} \right)^2 = \frac{\sum_{i=1}^{n} x_i^2}{\left(\sum x_i^2\right)^2} = \frac{1}{\sum_{i=1}^{n} x_i^2}$$

$$\Rightarrow \quad \therefore \quad \boxed{\sum_{i=1}^{n} k_i^2 = \frac{1}{\sum_{i=1}^{n} x_i^2}}$$

(3) $$\sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} \left(\frac{x_i}{\sum_{i=1}^{n} x_i^2} \right) x_i = \frac{\cancel{\sum_{i=1}^{n} x_i^2}}{\cancel{\sum_{i=1}^{n} x_i^2}} = 1$$

$$\sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} k_i (X_i - \bar{X}) = \sum_{i=1}^{n} k_i X_i - \underbrace{\cancel{\sum_{i=1}^{n} k_i}}_{=0} \bar{X}$$

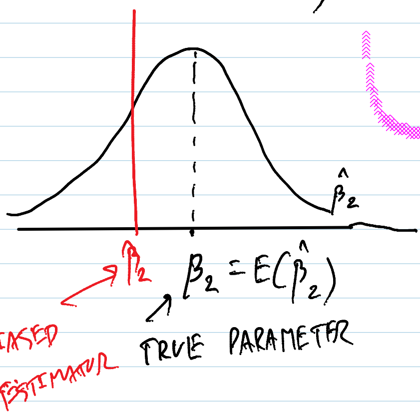
$$\boxed{\sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} k_i X_i = 1}$$

WE ARE GOING TO USE ALL PROPERTIES ABOVE TO DISCUSS ABOUT "UNBIASNESS OF OUR OLS ESTIMATORS" ($\hat{\beta}_1$, $\hat{\beta}_2$)

DISCUSS ABOUT 'UNBIASNESS OF OUR OLS ESTIMATORS $(\hat{\beta}_1, \hat{\beta}_2)$

IN STATISTICS, UNBIASNESS : $E(\hat{\beta}_1) = \beta_1$

$E(\hat{\beta}_2) = \beta_2$



7.9.2012

FROM $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = \bar{Y} - \sum_{i=1}^{n} k_i Y_i \bar{X}$

$$= \frac{\sum_{i=1}^{n} Y_i}{n} - \bar{X} \sum_{i=1}^{n} k_i Y_i$$

$$\hat{\beta}_1 = \sum_{i=1}^{n} \left[\frac{1}{n} - \bar{X} k_i \right] Y_i .$$

$$\hat{\beta}_1 = \sum_{i=1}^{n} \left[\frac{1}{n} - \bar{X} k_i \right] (\beta_1 + \beta_2 X_i + u_i)$$

$$= \sum_{i=1}^{n} \beta_1 \cdot \frac{1}{n} + \sum_{i=1}^{n} \beta_2 X_i \cdot \frac{1}{n} + \sum_{i=1}^{n} u_i \cdot \frac{1}{n} - \cancel{\bar{X} \sum_{i=1}^{n} k_i \beta_1}^{=0}$$

$$- \underbrace{\sum_{i=1}^{n} \beta_2 X_i \bar{X} k_i}_{- \sum_{i=1}^{n} k X_i \beta_2 \bar{X}, \ \sum k X_i = 1} - \sum_{i=1}^{n} \bar{X} k_i u_i .$$

$$= \beta_1 + \cancel{\beta_2 \bar{X}} - \cancel{\beta_2 \bar{X}} + \sum_{i=1}^{n} \left(\frac{1}{n} - \bar{X} k_i \right) u_i$$

$$\therefore \boxed{\hat{\beta}_1 = \beta_1 + \sum_{i=1}^{n} \left(\frac{1}{n} - \bar{X} k_i \right) u_i} .$$

TO CHECK UNBIASNESS, WE TAKE 'EXPECTATION' THROUGHOUT:

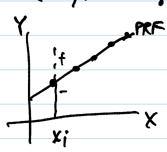
$$E(\hat{\beta}_1) = E(\beta_1) + E \underbrace{\left[\sum_{i=1}^{n} \left(\frac{1}{n} - \bar{X} k_i \right) u_i \right]}_{=0}$$

$$\boxed{E(\hat{\beta}_1) = \beta_1 .}$$

$=0$ BECAUSE X_i AND u_i ARE INDEPENDENT AND $E(u_i) = 0$.

(UNBIASNESS PROPERTIES

ON AVERAGE, MEAN VALUE OF $\hat{\beta}_1$ IS EQUAL TO THE TRUE β_1



HOW ABOUT $\hat{\beta}_2$? : IS IT ALSO UNBIASED ?

$$\Rightarrow \quad \hat{\beta}_2 = \sum_{i=1}^{n} k_i Y_i \quad \text{WHERE} \quad k_i = \frac{x_i}{\sum_{i=1}^{n} x_i^2}.$$

$$\hat{\beta}_2 = \sum_{i=1}^{n} k_i (\beta_1 + \beta_2 X_i + u_i)$$

$$= \beta_1 \underbrace{\sum_{i=1}^{n} k_i}_{=0} + \beta_2 \underbrace{\sum_{i=1}^{n} k_i X_i}_{=1} + \sum_{i=1}^{n} k_i u_i$$

$$= \beta_1 \cdot 0 + \beta_2 \cdot 1 + \sum_{i=1}^{n} k_i u_i$$

$$\hat{\beta}_2 = \beta_2 + \sum_{i=1}^{n} k_i u_i$$

TO CHECK UNBIASNESS, WE TAKE EXPECTATION THROUGHOUT:

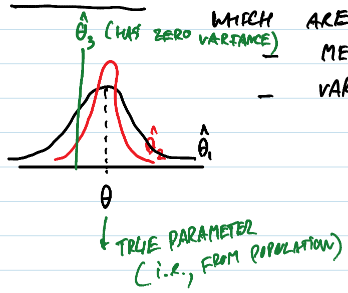
$$\boxed{E(\hat{\beta}_2) = \beta_2 .} \qquad \text{SINCE } E\left[\sum_{i=1}^{n} k_i u_i\right] = 0$$

THEREFORE, IN REPEATED EXPERIMENTS, MEAN OR AVERAGE VALUE OF OUR ESTIMATED COEFFICIENT, NAMELY, $E(\hat{\beta}_2)$ WILL BE EQUAL TO THE TRUE PARAMETER, β_2. [WE CALL THIS UNBIASNESS PROPERTY OF OUR OLS ESTIMATOR.]

SO FAR, WE DERIVED $\hat{\beta}_1$ AND $\hat{\beta}_2$.
WE ALSO PROVE THAT $E(\hat{\beta}_1) = \beta_1$ AND $E(\hat{\beta}_2) = \beta_2$.

NOW, WE WOULD LIKE TO ADDRESS ON THE ISSUE OF VARIANCE OF OUR ESTIMATOR.

BASIC IDEA: ANY DISTRIBUTION HAS BASIC CHARACTERISTICS WHICH ARE
- MEAN
- VARIANCE



ONCE YOU GET $\hat{\beta}_1$ AND $\hat{\beta}_2$ AND YOU SUCCESSFULLY PROVE THAT $E(\hat{\beta}_1) = \beta_1$ AND $E(\hat{\beta}_2) = \beta_2$. THEN NEXT TASK PEOPLE WILL ASK YOU ABOUT "THE DISTRIBUTION OF YOUR $\hat{\beta}_i$".

THEN $\text{VAR}(\hat{\beta}_1) = ?$ AND $\text{VAR}(\hat{\beta}_2) = ?$

- IS IT LARGE OR SMALL?
- HOW ABOUT ITS DISPERSION OR SPREAD AROUND MEAN?

$$\boxed{\text{var}(\hat{\beta}_1) = E[\hat{\beta}_1 - E(\hat{\beta}_1)]^2}$$

$E(\hat{\beta}_1)$: AVERAGE VALUE OF ALL POSSIBLE ESTIMATES

$$\boxed{\text{var}(\hat{\beta}_2) = E[\hat{\beta}_2 - E(\hat{\beta}_2)]^2}$$

NOTE: $\text{var}(\hat{\theta}) = E[\hat{\theta} - E(\hat{\theta})]^2$ (GENERAL FORM)

LET'S BEGIN W/ $\text{var}\,\hat{\beta}_2$ ...

2

LET'S BEGIN W/ $\mathrm{var}\,\hat{\beta}_2$ ...

$$\mathrm{var}(\hat{\beta}_2) = E\left[\hat{\beta}_2 - E(\hat{\beta}_2)\right]^2$$

$$= E\left[\hat{\beta}_2 - \beta_2\right]^2$$

REMEMBER THAT $\hat{\beta}_2 = \beta_2 + \sum_{i=1}^{n} k_i u_i$. (FROM ABOVE)

$$\hat{\beta}_2 - \beta_2 = \sum_{i=1}^{n} k_i u_i \,.$$

THEREFORE, $\mathrm{var}(\hat{\beta}_2) = E\left[\sum_{i=1}^{n} k_i u_i\right]^2$.

$$= E\left[\sum_{i=1}^{n} k_i u_i\right]\left[\sum_{i=1}^{n} k_i u_i\right]$$

$$= E\left[k_1u_1 + k_2u_2 + \dots + k_nu_n\right]\left[k_1u_1 + k_2u_2 + \dots + k_nu_n\right]$$

$$= E\left[k_1^2u_1^2 + k_2^2u_2^2 + \dots + k_n^2u_n^2 + \underbrace{2k_1u_1k_2u_2 + \dots}_{2k_iu_ik_ju_j}\right]$$

CROSS-PRODUCTS

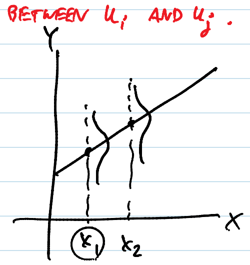
$$= k_1^2 E(u_1^2) + k_2^2 E(u_2^2) + \dots + k_n^2 E(u_2^2) + 0$$

$\because E(u_iu_j) = 0$

THERE IS NO ANY ASSOCIATION BETWEEN u_i AND u_j.

$$= k_1^2 \sigma_u^2 + k_2^2 \sigma_u^2 + \dots + k_n^2 \sigma_u^2$$

$$\boxed{\mathrm{var}(\hat{\beta}_2) = \sigma_u^2 \sum_{i=1}^{n} k_i^2 = \sigma_u^2 \frac{1}{\sum_{i=1}^{n} x_i^2}} \,.$$



SW → STATA 11

EVIEW 6

STAT TRANSFER