

Chapter 4 Inequalities without Absolute Values

We are all accustomed to using the sign for equality, [=] as used in the expression $3 \times 4 = 12$ to show that the quantities on the left and right of the sign are equal. However, in many practical situations we need a way of showing, not that two quantities are equal, but that one is bigger or smaller than another. For example, a course may have a limit of 50 students, so that the number of students recruited must be no more than 50; we may have advance orders for 120 of a certain product, so that number produced must be at least 120; and so on.

There are four signs used to represent these inequalities, as they are called:

- $a < b$ means “a is less than b”.
- $a > b$ means “a is greater than b”.
- $a \leq b$ means “a is less than or equal to b”.
- $a \geq b$ means “a is greater than or equal to b”.

In each case the larger end of the sign is directed towards the larger quantity. You are perhaps more likely to encounter the last two signs in practical applications. So we could express the statements in the last paragraph as “number of students ≤ 50 ” and “number of items produced ≥ 120 ”.

Definition 4.1: For any real numbers a and b :

- 1) a is greater than 0 if and only if a is positive real number, denoted by $a > 0$.
- 2) $a > b$ if and only if there exists a real number p such that $a = b + p$ or $a - b = p$.

For example, by Definition 4.1 part (2) we say that $5 > 1$ because $5 = 1 + 4$. In this case the real number p is 4.

Definition 4.2: For any real numbers a and b :

- 1) a is less than 0 if and only if a is negative real number and is denoted by $a < 0$.
- 2) $a < b$ if and only if $b > a$.

Properties of Real Numbers:

1. Closure property under addition: $a + b$ is real
Moreover, if $a > 0$ and $b > 0$, then $a + b > 0$
2. Closure proper under multiplication: ab is real
3. Commutative property under addition: $a + b = b + a$
4. Commutative property under multiplication: $ab = ba$
5. Associative property under addition: $(a + b) + c = a + (b + c)$
6. Associative property under multiplication: $(ab)c = a(bc)$
7. Distributive property: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$
8. $a < 0$ if and only if $-a > 0$
9. $a > 0$ if and only if $-a < 0$
10. $(-1)a = -a$
11. $-(-a) = a$
12. $(-a)b = a(-b) = -(ab)$
13. $(-a)(-b) = ab$
14. If $a > 0$ then $\frac{1}{a} > 0$
15. If $a < 0$ then $\frac{1}{a} < 0$
16. $a = b$ or $a < b$ or $a > b$ can be true only for one case.
17. $ab > 0$ if and only if $(a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$
18. $ab < 0$ if and only if $(a > 0$ and $b < 0)$ or $(a < 0$ and $b > 0)$

Definition 4.3: For any real numbers a and b :

- 1) a is greater than or equal to b if and only if $a > b$ or $a = b$, denoted by $a \geq b$.
- 2) a is less than or equal to b if and only if $a < b$ or $a = b$, denoted by $a \leq b$.

Definition 4.4: For any real numbers a and b :

- 1) a is not greater than b means $a \leq b$, denoted by $a \not> b$.
- 2) a is not less than b means $a \geq b$, denoted by $a \not< b$.

Section 4.1: Theorems about Inequalities without Absolute Values

There are several important theorems which we can use to solve the inequalities without absolute values.

Theorem 4.1.1: For any real numbers a and b :

- 1) $a \leq a$
- 2) $a \leq b$ and $b \leq a$ if and only if $a = b$

Proof:

Theorem 4.1.2: For any real numbers a , b , c , and d :

- 1) If $a < b$ and $b < c$, then $a < c$.
- 2) If $a < b$ then $a + c < b + c$.
- 3) If $a < b$ and $c < d$, then $a + c < b + d$.
- 4) If $a < b$ and $c > 0$, then $ac < bc$.
- 5) If $a < b$ and $c < 0$, then $ac > bc$.

Proof:

Theorem 4.1.3: For any real numbers a , b , c , and d :

- 1) If $a \leq b$ and $b \leq c$, then $a \leq c$.
- 2) If $a \leq b$ then $a + c \leq b + c$.
- 3) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.
- 4) If $a \leq b$ and $c > 0$, then $ac \leq bc$.
- 5) If $a \leq b$ and $c < 0$, then $ac \geq bc$.

Proof: They all are true by Theorem 4.1.2.

Theorem 4.1.4: For any real numbers a , b , c , and d where $0 < a < b$, $0 < c < d$, and n is any positive integer:

- 1) $0 < ac < bd$
- 2) $0 < a^n < b^n$
- 3) $\frac{1}{a} > \frac{1}{b} > 0$
- 4) $\frac{1}{a^n} > \frac{1}{b^n} > 0$

Proof:

Theorem 4.1.5: For any real numbers a , b , c , and d where $0 < a \leq b$, $0 < c \leq d$, and n is any positive integer:

1) $0 < ac \leq bd$

2) $0 < a^n \leq b^n$

3) $\frac{1}{a} \geq \frac{1}{b} > 0$

4) $\frac{1}{a^n} \geq \frac{1}{b^n} > 0$

Proof: They all are true by Theorem 4.1.4.

Theorem 4.1.6: For any real numbers a :

1) $a^2 \geq 0$

2) $a^2 = 0$ if and only if $a = 0$

Proof:

Theorem 4.1.7: For any real numbers a and b , if $a < b$ then $a < \frac{a+b}{2} < b$.

Proof:

Example 4.1.1: Let x and y be any positive real numbers, prove that if $x < y$ then $\frac{x+y}{y+1} < \frac{x+y}{x+1}$.

Example 4.1.2: Let x and y be any positive real numbers where $0 < x < y$, prove that

$$0 < \frac{2x}{y} < 1 + \frac{y^2}{x^2}.$$

Example 4.1.3: Let a and b be any positive real number where $a + 3 \leq b + 1 < 0$, prove that

$$2ab \leq ab^2 - a^2b.$$