

Solution: Exercise 3 (Part 2)

1. (a) Prove the statement:

“There are distinct integers m and n such that $\frac{1}{m} + \frac{3}{n}$ is an integer. ”

- (b) Disprove the statement: “ For all integers x and y , if $3x+y$ is odd then x and y are both odd. ”

Answer:

- (a) The statement:

“There are distinct integers m and n such that $\frac{1}{m} + \frac{3}{n}$ is an integer ” is an existential statement. So, we can prove this by give some specific values of m and n that make this statement true. Consider $m = 2$ and $n = 6$. Then

$$\frac{1}{m} + \frac{3}{n} = \frac{1}{2} + \frac{3}{6} = 1,$$

which is an integer. Therefore, the given statement is true. ■

(b) The statement “ For all integers x and y , if $3x + y$ is odd then x and y are both odd ” is a universal statement. We can **disprove** this statement by finding a counterexample that make this statement false or make its **negation** true. The negation is given by

“ there exist integers x and y , such that $3x + y$ is odd, and (but) x or y is even ”

Let $x = 2$ and $y = 1$. Notice that x is even and $3x + y = 3(2) + 1 = 5$ is odd. Hence, a counterexample is $x = 2$, $y = 1$ and we disprove the given statement. ■

2. Show that “for all integers m and n , if mn is even, then m is even or n is even,” by using

(a) a proof by contraposition,

(b) a proof by contradiction.

Answer:

(a)**Proof by contraposition:**

The contrapositive of the given statement is

“for all integers m and n , if m is odd and n is odd, then mn is odd.”

Let $m, n \in \mathbb{Z}$. Suppose m and n are odd. Then we can write

$m = 2k + 1$, for some $k \in \mathbb{Z}$ and

$n = 2s + 1$, for some $s \in \mathbb{Z}$

Consider mn :

$$mn = (2k + 1)(2s + 1) = 4ks + 2k + 2s + 1 = 2 \underbrace{(2ks + k + s)}_{=:r \in \mathbb{Z}} + 1 = 2r + 1,$$

which is an odd integer, by definition. Note that $r = 2ks + k + s$ is an integer since it is the sum of product of integers $k, s \in \mathbb{Z}$.

The contrapositive is equivalent to the original statement and therefore showing that its contrapositive is true implies that the original statement is also true. ■

(b) **Proof by contradiction:**

Suppose not. I.e. suppose that its negation is true:

“there exist integers m and n , such that mn is even, and m, n are both odd.”

Then since m, n are both odd, we can write $m = 2k + 1$, for some $k \in \mathbb{Z}$ and $n = 2s + 1$, for some $s \in \mathbb{Z}$

Consider mn :

$$mn = (2k + 1)(2s + 1) = 4ks + 2k + 2s + 1 = 2 \underbrace{(2ks + k + s)}_{=:r \in \mathbb{Z}} + 1 = 2r + 1,$$

which is an odd integer, by definition, since $r = 2ks + k + s$ is an integer.

This contradicts to the assumption that mn is even. Hence its negation is false and the statement itself is true. ■

3. Prove by the **method of exhaustion** that “ $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$.”

Answer Given that an integer n such that $1 \leq n \leq 4$ implies that $n = 1, 2, 3$, or 4 .

For $n = 1$, $1^2 + 1 = 2^1$ and $n^2 + 1 \geq 2^n$ is true.

For $n = 2$, $2^2 + 1 = 5$ and $2^2 = 4$, so $n^2 + 1 \geq 2^n$ is true.

For $n = 3$, $3^2 + 1 = 10$ and $2^3 = 8$, so $n^2 + 1 \geq 2^n$ is true.

For $n = 4$, $4^2 + 1 = 17$ and $2^4 = 16$, so $n^2 + 1 \geq 2^n$ is true.

Therefore, $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$. ■

4. Use the **proof by cases** to show that “Prove that for all integers m and n , $m + n$ and $m - n$ are either both odd or both even.”

[Hint: Consider 4 cases of even and odd for m and n]

Answer: We will consider 4 cases for m and n .

- Case I: m is even and n is even.

That is, $m = 2k$ and $n = 2s$ for some integers $k, s \in \mathbb{Z}$. which implies $m - n$ is even because $k - s$ is an integer.

$$m + n = 2k + 2s = 2(k + s)$$

Then, which implies $m + n$ is even because $k + s$ is an integer. Also,

$$m - n = 2k - 2s = 2(k - s)$$

Hence, both $m + n$ and $m - n$ are both even in this case.

- Case II: m is even and n is odd.

That is, $m = 2k$ and $n = 2a + 1$ for some integers $k, s \in \mathbb{Z}$. Then,

$$m + n = 2k + (2a + 1) = 2(k + a) + 1$$

which implies $m + n$ is odd because $k + a$ is an integer. Also,

$$m - n = 2k - (2a + 1) = 2(k - a) - 1 = 2(k - a) - 1 + 2 - 2 = 2(k - a - 1) + 1$$

which implies $m - n$ is odd because $k - a - 1$ is an integer. Hence, both $m + n$ and $m - n$ are both odd in this case.

- Case III: m is odd and n is even. That is, $m = 2b + 1$ and $n = 2s$ for some integers $k, s \in \mathbb{Z}$. Then,

$$m + n = (2b + 1) + 2s = 2(b + s) + 1$$

which implies $m + n$ is odd because $b + s$ is an integer. Also,

$$m - n = (2b + 1) - 2s = 2(b - s) + 1$$

which implies $m - n$ is odd because $b - s$ is an integer. Hence, both $m + n$ and $m - n$ are both odd in this case.

- Case IV: m is odd and n is odd.

That is, $m = 2b + 1$ and $n = 2a + 1$ for some integers $k, s \in \mathbb{Z}$. Then,

$$m + n = (2b + 1) + (2a + 1) = 2(b + a + 1)$$

which implies $m + n$ is even, by definition, because $b + a + 1$ is an integer. Also,

$$m - n = (2b + 1) - (2a + 1) = 2(b - a)$$

which implies $m - n$ is even, by definition, because $b - a$ is an integer. Hence, both $m + n$ and $m - n$ are both even in this case. ■

5. Consider the statement: for any integer $n \geq 0$,

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^n = \frac{1}{2}(5^{n+1} - 1).$$

Suppose we want to prove the above statement by **mathematical induction**.

- (a) What is $P(n)$?
- (b) Write $P(0)$: Is $P(0)$ true?
- (c) Write $P(k)$:
- (d) Write $P(k + 1)$:
- (e) Prove the above statement: $\sum_{j=0}^n 2 \cdot 5^j = \frac{1}{2}(5^{n+1} - 1)$ by using **mathematical induction**.

Answer:

- (a) What is $P(n)$?
 $P(n)$ is the statement

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^n = \frac{1}{2}(5^{n+1} - 1).$$

- (b) Write $P(0)$: Is $P(0)$ true?
 $P(0) : 2 = \frac{1}{2}(5^{0+1} - 1) = 2$ is true.

- (c) Write $P(k)$:

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^k = \frac{1}{2}(5^{k+1} - 1).$$

- (d) Write $P(k+1)$:

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^{k+1} = \frac{1}{2}(5^{k+2} - 1).$$

- (e) Prove the above statement: $\sum_{j=0}^n 2 \cdot 5^j = \frac{1}{2}(5^{n+1} - 1)$ by using **mathematical induction**.

We want to prove that $P(n)$ is true for all integer $n \geq 0$.

(I) **Basis step:** $P(0) : 2 = \frac{1}{2}(5^{0+1} - 1) = 2$ is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 0$.

Assume that $P(k)$ is true:

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^k = \frac{1}{2}(5^{k+1} - 1). \quad \text{—————}(\star) \text{ “inductive hypothesis”}$$

We want to show that $P(k+1)$ is true:

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^{k+1} = \frac{1}{2}(5^{k+2} - 1).$$

$$\begin{aligned} \underbrace{2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^k}_{(\star)} + 2 \cdot 5^{k+1} &= \frac{1}{2}(5^{k+1} - 1) + 2 \cdot 5^{k+1} \text{ by } (\star) \\ &= 5^{k+1} \underbrace{\left(\frac{1}{2} + 2 \right)}_{5/2} - \frac{1}{2} \\ &= \frac{5^{k+2}}{2} - \frac{1}{2} \\ &= \frac{1}{2}(5^{k+2} - 1), \end{aligned}$$

which implies that $P(k+1)$ is true. Hence, from (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 0$ by the induction proof. ■

6. Use mathematical induction proof to show that

$$2^n > n^2$$

for an integer greater than 4.

Answer:

Let $P(n)$ be the statement $2^n > n^2$.

We want to prove that $P(n)$ is true for all integer $n \geq 5$.

(I) **Basis step:** $P(5)$: $2^5 > 5^2$ or $32 > 25$, which is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 5$.

Assume that $P(k)$ is true:

$P(k)$: $2^k > k^2$. —————(★) “inductive hypothesis”

We want to show that $P(k+1)$ is true: $2^{k+1} > (k+1)^2$.

Note that for $k \geq 5$, we have $(k-2) \geq 3$ and $k(k-2) \geq 15 > 1$:

$$\begin{aligned} k(k-2) &> 1 \\ k^2 - 2k &> 1 \\ k^2 + k^2 - 2k &> k^2 + 1 \\ 2k^2 &> k^2 + 2k + 1 = (k+1)^2. \end{aligned}$$

That is, $2k^2 > (k+1)^2$ for any integer $k \geq 5$. —————(♠)

Consider 2^{k+1} from $P(k+1)$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^2 && \text{by “inductive hypothesis” (★) } 2^k > k^2 \\ &> (k+1)^2 && \text{by (♠) } 2k^2 > (k+1)^2 \end{aligned}$$

which implies that $P(k+1)$ is true.

Hence, from (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 0$ by the induction proof. ■

Remark: To show (♠), it is also possible to use the proof by induction (see next page).

Note that to show ($\spadesuit$), it is also possible to use the proof by induction. Let $Q(n)$ be the statement $\boxed{2n^2 > (n+1)^2}$.

We want to prove that $Q(n)$ is true for all integer $n \geq 5$.

(i) **Basis step:** $Q(5) : 2 \cdot 5^2 > (5+1)^2$ or $50 > 36$, which is true.

(ii) **Inductive step:** Assume that $Q(k) : 2k^2 > (k+1)^2$ — (★★).

We want to show that $Q(k+1) : 2(k+1)^2 > (k+2)^2$.

$$\begin{aligned}
 2(k+1)^2 &= 2k^2 + 4k + 2 \\
 &> (k+1)^2 + 4k + 2 && \text{by “inductive hypothesis” (★★) } 2^k > k^2 \\
 &= k^2 + 2k + 1 + 4k + 2 \\
 &= (k^2 + 4k + 4) + \underbrace{2k - 1} \\
 &= (k+2)^2 + \underbrace{2k - 1}_{\geq 0} \\
 &> (k+2)^2 && \text{since } 2k - 1 > 0 \text{ for } k \geq 5
 \end{aligned}$$

which implies that $2(k+1)^2 > (k+2)^2$ and $P(k+1)$ is true.

From (i) and (ii), $\boxed{2n^2 > (n+1)^2}$ for all $n \geq 5$.

7. (Optional) Use the method of constructive proof to show that:

if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

Answer: Constructive proof

Let $r, s \in \mathbb{R}$ such that $r < s$. Let

$$x = \frac{r+s}{2}.$$

We will show that for this particular x has the value between the r and s .

$$\begin{array}{rcl}
 r & < & s \\
 r+r & < & s+r \\
 \frac{r+r}{2} & < & \frac{s+r}{2} \quad \Rightarrow \quad r < x \\
 \underbrace{\hspace{1cm}}_{=r} & & \underbrace{\hspace{1cm}}_{=x}
 \end{array}$$

$$\begin{array}{rcl}
 r & < & s \\
 r+s & < & s+s \\
 \frac{r+s}{2} & < & \frac{s+s}{2} \quad \Rightarrow \quad x < s \\
 \underbrace{\hspace{1cm}}_{=x} & & \underbrace{\hspace{1cm}}_{=s}
 \end{array}$$

That is, for any given r and s , we can always find $x = \frac{r+s}{2}$ such that $r < x < s$.

Note that it is also possible to use a different value of x . ■

