# Chapter 5

## Nonlinear Model and Differential Calculus in Economic Theory

*Topics:*

- Quadratic theory
- Other nonlinear functions
- Slope and derivatives of a function
- Rule of differentiation
- Non differentiable functions
- Convexity and concavity
- Maxima-Minima
- Inflection point
- Examples in Economics
    - ➲ Derivative and marginality
    - ➲ Relations among the total, the average and the marginal functions
    - ➲ Elasticity, total revenue and marginal revenue

## Comparative Statics and Derivative

***VOCAB***: "Difference Quotient" / "Derivative"/ "Differentiation"/ "Differential Calculus"

Comparative Statics is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables.

For example,

When $G = G_0$

$$\left.\begin{aligned}&(1.)\ Y = C + I + G\\&(2.)\ C = a + bY_d\\&(3.)\ Y_d = Y - T\\&(4.)\ I = I_0\\&(5.)\ G = G_0\end{aligned}\right\}\Rightarrow Y_E\big|_{G=G_0} = \frac{a - bT + I_0 + G_0}{1-b}$$

When $G = G_1$

$$\left.\begin{aligned}&(1.)\ Y = C + I + G\\&(2.)\ C = a + bY_d\\&(3.)\ Y_d = Y - T\\&(4.)\ I = I_0\\&(5.)\ G = G_1\end{aligned}\right\}\Rightarrow Y_E\big|_{G=G_1} = \frac{a - bT + I_0 + G_1}{1-b}$$

$$\begin{aligned}&when \quad G_0 \quad \to \quad Y_E\big|_{G_0}\\&when \quad G_1 \quad \to \quad Y_E\big|_{G_1}\end{aligned}$$

$$\Delta G = G_1 - G_0 \quad \Rightarrow \quad \Delta Y = Y_E\big|_{G_1} - Y_E\big|_{G_0}$$

It should be clear that the problem under consideration is essentially one of finding *a rate of change: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular parameter or exogenous variable.*

The notion of rate of change is directly concerned with the mathematical concept of *derivative,* in *differential calculus*.

Looking at a function: $y = f(x)$

$$\begin{array}{llll} when & x_1 & \rightarrow & y_1 = f(x_1) \\ when & x_2 & \rightarrow & y_2 = f(x_2) \end{array}$$

$$\Delta x = x_2 - x_1 \quad \Rightarrow \quad \Delta y = y_2 - y_1$$

We are interested in ……………which is called "……………………………………"

When $\Delta x \rightarrow 0$, ……………………………………………………………………

, which is called "……………………………………."

Notation for derivative of $f$ at $x$: $f'(x)$, $\frac{df(x)}{dx}$, $\frac{dy}{dx}$, $D_x f(x)$, $y'$.

A *derivative* is a function. The word *derivative* means a derived function, from the original function $y = f(x)$, which is also called a *primitive function*.

When we have a Primitive function $y = f(x)$ and try to find Derived function $\frac{dy}{dx}$, we call this process "differentiation".

Derivative and differential Calculus will be used in finding maximum, minimum points and in optimization problem.

## The Slope of a Curve and The Derivative of a function

The concept of the **slope** of a curve is the geometric counterpart of the concept of the **derivative**. Both concepts deal with the **marginal** notion used in economics.

The slope of a total cost function: $C = F(q)$ measures the change in total cost resulting from a unit increase in output, i.e. the marginal cost(MC). That is,

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The slope of a utility function: $u = U(x)$ measures the change in utility resulting from a unit increase in consumption, i.e. the marginal utility(MU). That is,

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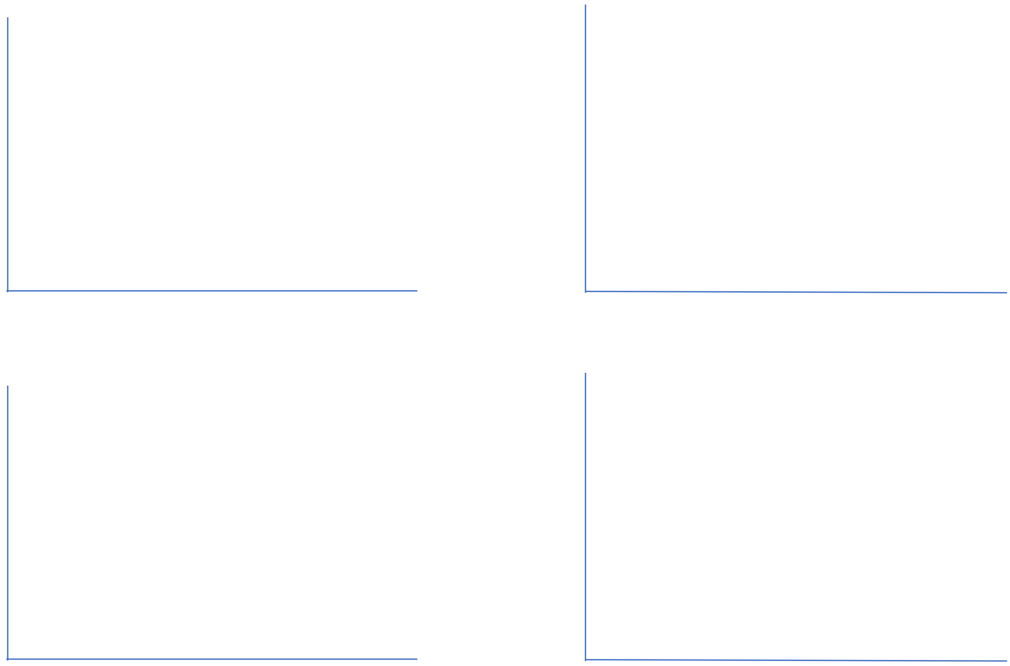
⊙ If a function is linear, the slope is constant and is equal for every points on the linear curve.

⊙ If a function is nonlinear function, the slope is not constant. Slope for each point on the curve might not be equal. Slope at each point on a nonlinear function is the slope of the tangent line at that point.



**Differentiation / derivation / process of obtaining the derivative**

From definition of derivative, we can find the derivative from a primitive function as the following.

**For example,** let $f(x) = 2x^2 + 4$ find $f'(x)$:

(1.) $f'(x) = 4x$

(2.) Use derivative definition: $f'(x) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$

Let $x$ increases from $x_1$ to $x_2 = x_1 + h$, so $\Delta x = x_2 - x_1 = h$ with

$$f(x_1) = 2x_1^2 + 4 \text{ และ } f(x_2) = 2x_2^2 + 4 = 2(x_1 + h)^2 + 4 = 2x_1^2 + 4x_1h + 2h^2 + 4$$

Therefore,

$$\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{f(x_2)-f(x_1)}{h}$$
$$=\frac{\left(2x_1^2+4x_1h+2h^2+4\right)-\left(2x_1^2+4\right)}{h}$$
$$=4x_1+2h$$

$x_1$ is actually can be any $x$ $\frac{\Delta y}{\Delta x}=4x+2h$

$$f'(x)=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)$$
$$=\lim_{h\to 0}(4x+2h)$$
$$=4x \qquad \#$$

**Differentiability of a function**

"A function is differentiable at point $x_0$ if it is smooth and continuous at point $x_0$."

**✪ Continuity**

🔔 Continuity is a necessary condition for a function to be differentiable.

The function $f$ is continuous at $a$ if

1. We can find $f(a)$, i.e. $x=a$ must be in the domain of the function $f$.
2. We can find $\lim_{x\to a} f(x)$
3. $\lim_{x\to a} f(x)=f(a)$

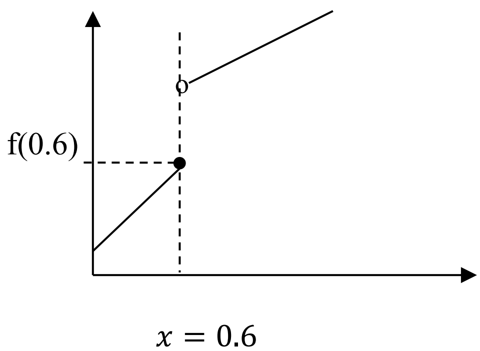
**Example of a function that is not continuous at** $x=0.6$

$f(x)$

เนื่องจาก

1. $f(0.6)$ can be found.
2. $\lim_{x\to 0.6^-} f(x)\neq \lim_{x\to 0.6^+} f(x)$

$\therefore f(x)$ is not continuous at $x=0.6$



H.w.: Is this function continuous $f(x)=\begin{cases} x^2 & when\ x<2 \\ x+1 & when\ x\geq 2 \end{cases}$ ?

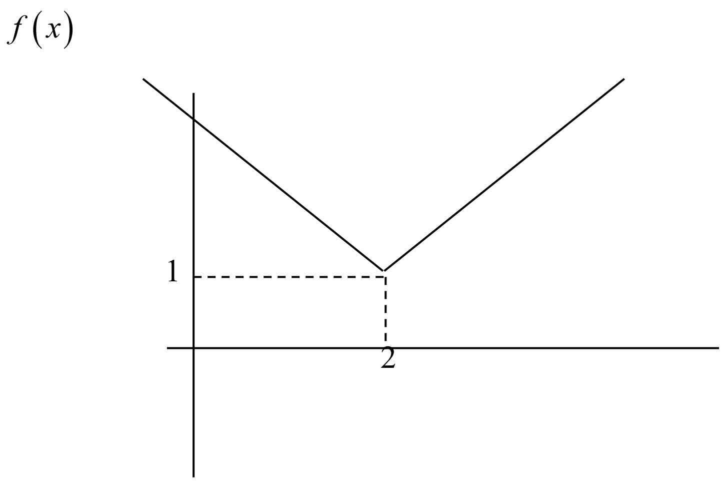
## ✪ Smooth Function(has no kink)

🔔 Smooth function is a sufficient condition for differentiability.

The differentiability condition is:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

Example of a function that has kink: $f(x)=|x-2|+1$




Continuity: $f(2)=\lim_{x\to 2} f(x)=1$, $f(x)$ is continuous at $x=2$

But if we try to find derivative:

Function $f$ has derivative at $a$ if:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

If we let $x=a+h$

when $h\to 0$ , $x\to a$ and we can rewrite $f'(a)$ as:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad , \quad x\neq a$$

Derivative of $f(x)$ at $x=2$ is:

$$f'(2)=\lim_{x\to 2}\frac{|x-2|+1-1}{x-2}=\lim_{x\to 2}\frac{|x-2|}{x-2}$$

$$where\ \lim_{x\to 2^-}\frac{|x-2|}{x-2}=\frac{-(x-2)}{x-2}=-1$$

$$\lim_{x\to 2^+}\frac{|x-2|}{x-2}=\frac{x-2}{x-2}=1$$

$$\therefore \lim_{x\to 2^-}\frac{|x-2|}{x-2}\neq \lim_{x\to 2^+}\frac{|x-2|}{x-2}$$

Therefore, $f'(2)=\lim_{x\to 2}\frac{|x-2|}{x-2}$ cannot be found.

Even $f(x)$ is continuous at $x=2$ but we cannot find $f'(2)$ because $f(x)$ has a kink at $x = 2$.

**Notation:**

$f \in c^{(0)}$ or $f \in c$ : means $f$ is continuous.

$f \in c^{(1)}$ or $f \in c'$ : means $f$ is continuously differentiable (A function $f$ with a continuous derivative function, i.e. the everywhere-smooth function)

**Rule of Differentiation**

**1.** If $f(x) = c$, $c$ is a constant $f'(x) = 0$

**2.** If $f(x) = cg(x)$, $c$ is a constant $f'(x) = cg'(x)$

**3.** If $f(x) = x^n$, $n$ is any real number $f'(x) = nx^{n-1}$

**4.** If $f(x) = U(x) \pm V(x)$, $f'(x) = U'(x) \pm V'(x)$

**5.** If $f(x) = U(x)V(x)$, $f'(x) = U(x)V'(x) + V(x)U'(x)$

**6.** If $f(x) = \frac{U(x)}{V(x)}$, $f'(x) = \frac{V(x)U'(x)-U(x)V'(x)}{[V(x)]^2}$

**7.** [chain rule] If we have a differentiable function $z = U(y)$ and another differentiable function $y = V(x)$ , then the derivative of $z$ with respect to $x$ is equal to the derivative of $z$ with respect to $y$, times the derivative of $y$ with respect to $x$ .

$$\frac{dz}{dx}=\frac{dz}{dy}\frac{dy}{dx}=U'(y)V'(x)$$

Change in $x$ determines change in $y$ via function $V$, and change in $y$ determines change in $z$ via function $U$.

**8.** [Derivatives of Inverse Function] Let $y = f(x)$, we have $\frac{dy}{dx}$. The inverse function of $f, x = f^{-1}(y)$, its derivative is $\frac{dx}{dy}$, and

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

**9.** [Derivatives of log function]
If $y = \log_a x$, $\frac{dy}{dx} = \frac{1}{xlna}$.
If $y = ln\,x$, $\frac{dy}{dx} = \frac{1}{xlne} = \frac{1}{x}$.
If $y = ln\,V(x)$, $\frac{dy}{dx} = \frac{V'(x)}{V(x)}$.

**10.** [Derivatives of exponential function]

If $y = a^x$, where $a > 0, a \neq 1$, $\frac{dy}{dx} = a^x lna$

If $y = e^x$ , $\frac{dy}{dx} = e^x$

If $y = e^{V(x)}$, $\frac{dy}{dx} = e^{V(x)}V'(x)$

H.w.:

(a.) find $f'(x)$ for the following functions:

f(x) = $\sqrt{2}x^{\frac{-2}{3}}$

f(x) = $2x^3 + 3x^2 - 5x + 1$

$f(x) = (2x+3)(3x^2)$

$f(x) = \frac{ax^2 + b}{cx}$

$f(x) = (x - 4x^2)^3$

$f(x) = (1 - x^2)\sqrt{1 - 2x}$

(b.)

If $y = f(x) = 5x + 25$, find $\frac{dx}{dy}$

If $y = f(x) = x^5 + x$, find $\frac{dx}{dy}$

(c.)

Find $f'(x)$ if $f(x) = e^{3x-1}$

Find $f'(x)$ if $f(x) = \ln(1 + 2x + x^2)$

Find $f'(x)$ if $f(x) = (1 + x)(1 + e^{x^2})(3 - x)^{\frac{1}{2}}$

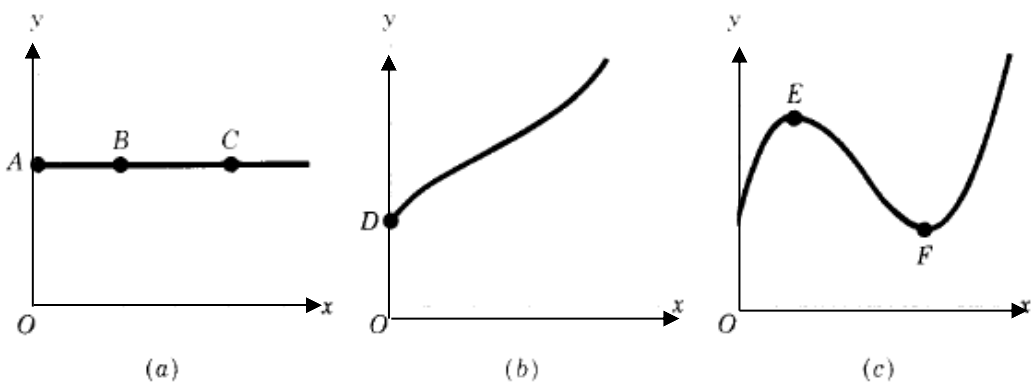
Find $f'(x)$ if $f(x) = (x^2 - 1)\sqrt{\frac{2 - 3x^2}{1 - 2x^3}}$

(d.) Let $y = x^4 - x^{\frac{4}{3}} + 6x^{\frac{1}{3}}$, find the second derivative of $f(x)$ with respect to $x$, $f''(x)$.

**Maxima and Minima, Convexity and Concavity**

**Global vs. Local Extremum Concept**

Consider (a), (b) , (c) and (d)



**Graph(a.):** Constant Function

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**Graph (b.):** Strictly Increasing Function

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*Note:*

Monotonic Increasing Function

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Monotonic Decreasing Function

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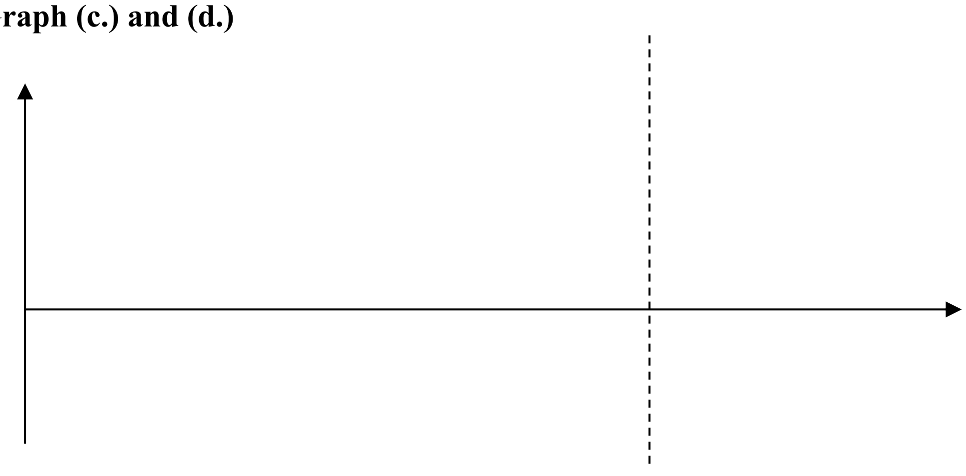
**Graph (c.) and (d.)**



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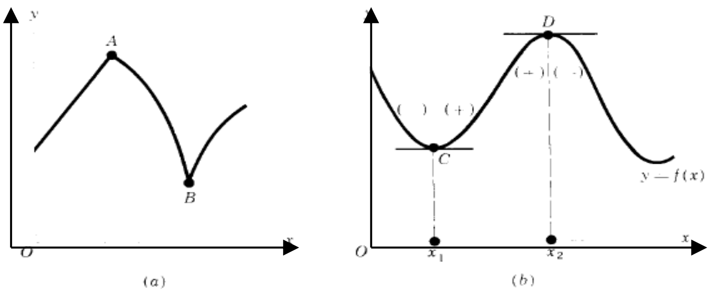
**✪ First-Derivative Test ✪**

If y = f(x) has Local Max/ Local Min) at $x = x_0$, possible cases for the first derivative are:

1. $f'(x_0)$ *does not exist*

2.

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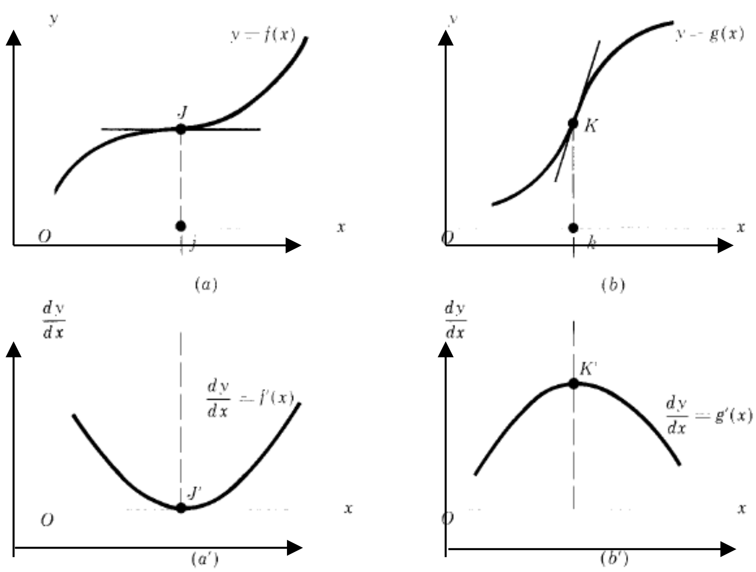
|---|
| If $f(x)$ **is a smooth function, the necessary condition** for a point to be a local max or local min is: |

We call the order pair $(x, y)$ that satisfies this necessary condition a stationary point, in which $x$ is “critical point or critical value”, and $y$ is “stationary value”.

The condition $f'(x) = 0$ is a necessary condition, but not a sufficient condition) for a point to be a local maximum or local minimum.



If the first derivative of f(x) at $x = x_0$ is 0 ($f'(x_0) = 0$), then $f(x_0)$ will be

1. **Local maximum,** when the first derivative changes from being positive to negative when $x < x_0$ and $x > x_0$

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2. **Local minimum,** when the first derivative changes from being negative to positive when $x < x_0$ and $x > x_0$

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3. **Inflection point,** when the first derivative doesn't change sign when $x < x_0$ and $x > x_0$

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H.w.: Find local maxima or minima of the following functions:

$y = f(x) = x^3 - 12x^2 + 36x + 8$

Average-Cost Function $AC = f(Q) = Q^2 - 5Q + 8$

**✪ Second-Derivative Test ✪**

Interpretation of the Second Derivative

$f'(x)$ ……………………………………………………

$f''(x)$ ……………………………………………………

$\left.\begin{array}{l} f'(x) > 0 \\ f'(x) < 0 \end{array}\right\}$ f(x) will be $\Big\}$

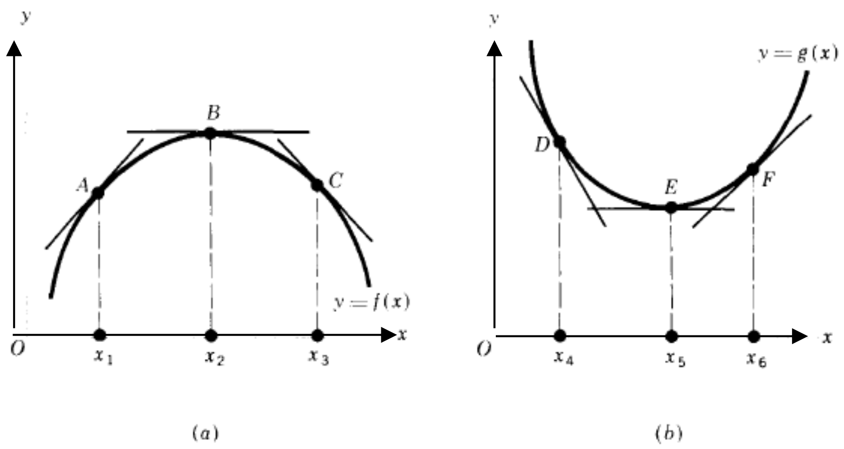
$\left.\begin{array}{l} f''(x) > 0 \\ f''(x) < 0 \end{array}\right\}$ ……………f(x) will be $\Big\}$

If $f'(x) > 0$ and $f''(x) > 0$,

If $f'(x) > 0$ and $f''(x) < 0$,

If $f'(x) < 0$ and $f''(x) > 0$,

If $f'(x) < 0$ and $f''(x) < 0$,




From graph (a.)

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From graph (b.)

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If $f'(x_0) = 0$, $f(x_0)$ will be

→ a local maximum when……………………………………………………………

→ a local minimum when……………………………………………………………

**Curvature of a Graph**

A graph can be:

→ Strictly Convex/ Convex

→ Strictly Concave/Concave

$f(x)$ is strictly convex if a linear line between any two points M and N lies above $f(x)$

$f''(x)$…………………. for strictly convex function

$f(x)$ is strictly concave if a linear line between any two points M and N lies below $f(x)$

$f''(x)$…………………. for strictly concave function

SUMMARY

| Condition | Maximum | Minimum |
|---|---|---|
| **First-order necessary** | | |
| **Second-order necessary** | | |
| **Second-order sufficient** | | |

*H.W.:*

(1.) $f(x)=\frac{1}{8}\left(x^4-8x^2\right)$

(2.) $f(x)=\frac{x^4}{4}-\frac{3}{2}x^2$

(3.) $f(x)=x^4$

## Application of differential calculus in economics

**Demand** **(1.)** $Q^d = a + bP$

**I.** $\dfrac{dQ^d}{dP}$

**II.** $\dfrac{d^2Q^d}{dP^2}$

The graphs are:

**(2.)** $Q^d = \dfrac{a}{P}$

## Total Utility

$$\frac{dTU}{dQ} > 0$$

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$$\frac{d^2TU}{dQ^2} < 0$$

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**Total Utility**

$\frac{dy}{dx}$ **and marginality**

$TC = 5Q^2 + 3Q + 100 \quad \rightarrow "Total\ cost"$

$TR = 25Q \quad \rightarrow "Total\ revenue"$

$U = 5x^{1/2} + 100 \quad \rightarrow "Utility\ function"$

**MC =**

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**MR=**

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**MU=**

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$\frac{dy}{dx}$ and price elasticity of demand: law of demand

***Demand*** **(1.)** $Q^d = 250 - 10P$ **(2.)** $Q^d = 50P^{-1/4}$

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- $\frac{dy}{dx}$ and income elasticity of demand: inferior vs. normal good

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- $\frac{dy}{dx}$ and cross price elasticity of demand: substitute vs. complementary goods

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- $\frac{dy}{dx}$ and law of supply

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- $\frac{dy}{dx}$ and output elasticity of labor

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## Total Cost, Average Cost, and Marginal Cost

Relationship between MC and AC

How much does total cost increase if labor in production increases?

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**TR, AR, MR**

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**The relation between MR, AR, and Price elasticity of demand**

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The elasticity of total revenue with respect to output

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How much does total revenue increase if labor in production increases?

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H.W. Let $P = 25 - 0.1Q$, $Q = 5L$, find MRP

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**TP, $AP_L$, $MP_L$**

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Proof:

1.) At the maximum of $AP_L$, $AP_L = MP_L$

2.) When $MP_L > AP_L$, $AP_L$ increases.

3.) When $MP_L < AP_L$, $AP_L$ decreases.

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$MC$ and $MP_L$

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✍ $AVC$ and $AP_L$

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