

Chapter 3

Calculus of Single Variable: Applications

3.1 Labor Union In a given industry, the labor union is effectively the monopoly of labor input. The market demand for the labor is a negatively sloped as shown in the figure below.

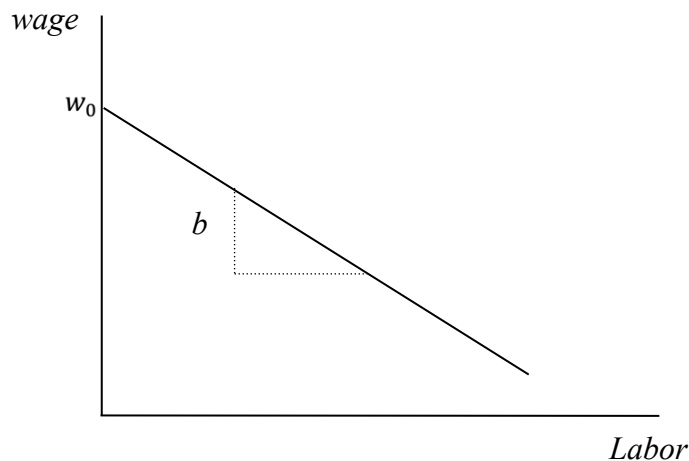


Figure 3.1 Demand for union workers

The demand for union workers is given by,

$$w = w_0 - bL$$

where w is the wage earned by union members in a given industry employing L union workers.

There are 4 possible goals the union can have:

1. max wage rate
2. max level of employment
3. max total income of employed workers
4. max rents earned by employed workers

1) Max Wage Rate: Optimal solution $w^* = w_0 = w_1$, and so $L = 0$ and it is a corner solution—a solution where the sufficient conditions are not met.

2) Max Employment: Let

$\bar{w}$ = wages earned by union members working in other industry or receiving unemployment compensation.

Then, optimal solution $w^* = \bar{w} = w_2$. That is, the union will set the wage at the level the worker can get at the other best paying industry and $\bar{L}$ members will be hired.

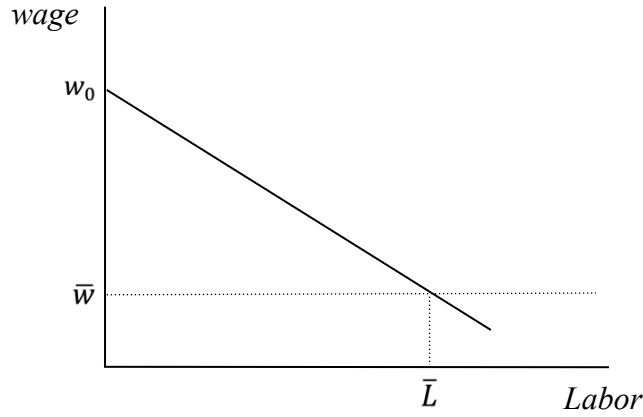


Figure 3.2 The best wage rate to set if the union maximizes employment

3) Max Total Income of Employed Union Workers:

Let I be the total income earned by all employed union workers. We can see that the total income depends on wage rate w , and so we write $I = I(w)$ where,

$$\begin{aligned} I(w) &= wL \\ &= \frac{w(w_0 - w)}{b}, w = w_0 - bL. \end{aligned}$$

By the first-order sufficient condition,

$$I'(w^*) = \frac{w_0 - 2w^*}{b} = 0,$$

and we have the critical point $w^* = \frac{w_0}{2}$. We can test that this critical point satisfies the second-order sufficient condition

$$I''(w^*) = -\frac{2}{b} < 0.$$

So the optimal wage rate to set is $w_3 = \max\{\frac{w_0}{2}, \bar{w}\}$.

4) Max Economic Rent: We define rent as

$$r = w - \bar{w},$$

i.e., the difference between the wage received and the highest wage earned in other industry. The total rent earned is

$$R(w) = rL = (w - \bar{w}) \frac{w_0 - w}{b},$$

first-order sufficient condition,

$$R'(w^*) = \frac{\bar{w} + w_0 - 2w^*}{b} = 0$$

and the critical point is

$$w^* = \frac{\bar{w} + w_0}{2}.$$

The second-order sufficient condition is given by

$$R''(w^*) = -\frac{2}{b} < 0.$$

We will have the optimal solution

$$w_4 = \frac{\bar{w} + w_0}{2} = w_3 + \frac{\bar{w}}{2}, \text{ if } w_3 > \bar{w}.$$

- Are w_3 and w_4 strict global maximum points?

In summary, we can draw a graph to show the relation of optimal wage rates of four cases as

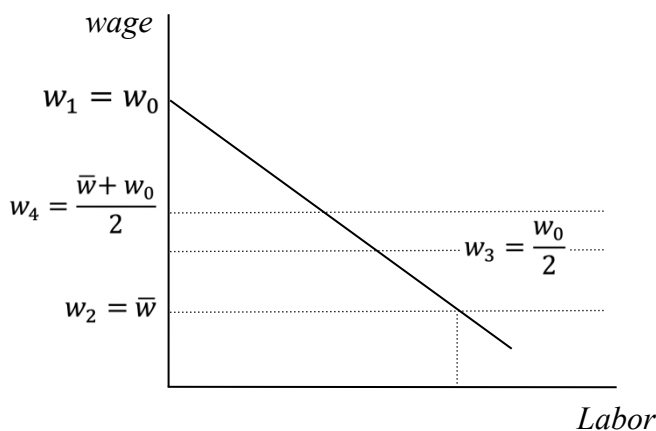


Figure 3.3 Optimal wages for 4 different objectives

HW Maximize contribution from the union workers to the union if the contribution is

- \$X per worker
- $x\%$ of the wage earned by each worker

HW: p. 61 #2.1, 2.2, 2.3

3.2 Profit Maximization in Perfect Competition

A competitive firm with production function

$$Q = f(L, K) = 2L^{0.5}K^{0.5}.$$

Assume short-run production with $K = K_0$. The profit now has only L as decision variable.

$$\begin{aligned}\max \pi(L) &= pQ - wL - rK_0 \\ &= p2L^{0.5}K_0^{0.5} - wL - rK_0.\end{aligned}$$

By first-order sufficient condition,

$$\pi'(L^*) = pL^{*0.5}K_0^{0.5} - w = 0,$$

we have the critical point

$$L^* = \frac{p^2 K_0}{w^2},$$

independent of r . This can be considered the short-run demand for labor.

This critical point is the maximum point because it satisfies the second-order sufficient condition

$$\pi''(L^*) = -0.5pL^{*-1.5}K_0^{0.5} < 0.$$

The optimal output is $Q = 2L^{*0.5}K_0^{0.5} = \frac{2pK_0}{w}$, which is a strict global maximum point.

HW Compute the demand elasticity of the short run demand for labor $L^* = \frac{p^2 K_0}{w^2}$.

HW Redo this example with $Q = AL^\alpha K_0^\beta$. Is there any condition on the values of α and/or β ?

HW Baldani, p. 62, #2.4.

3.3 Profit Maximization of a Monopoly A monopoly faces the demand function

$$p = a - bQ,$$

and has the total cost function

$$TC = f + cQ.$$

We have,

$$\begin{aligned}\pi(Q) &= TR - TC = (a - bQ)Q - f - cQ \\ \pi'(Q^*) &= MR - MC = a - 2bQ^* - c = 0 \\ \pi''(Q^*) &= -2b < 0\end{aligned}$$

$$Q^* = \frac{a - c}{2b}$$
$$p^* = a - bQ^* = \frac{a + c}{2}.$$

Note the optimal solution is independent of the fixed cost f .

HW Baldani, p.62, #2.6, 2.7

3.4 Taxation on Monopoly Specific tax $\$t$ per unit sold is imposed on a monopoly. The tax collected is given by

$$T = tQ.$$

The profit of the monopoly, assuming a linear demand function, is given by

$$\begin{aligned}\pi(Q) &= TR - TC - T = (a - bQ)Q - f - cQ - tQ \\ &= (a - bQ)Q - (c + t)Q - f.\end{aligned}$$

Maximizing this profit, the first- and second-order sufficient conditions are given by

$$\begin{aligned}\pi'(Q^*) &= MR - MC - t = a - 2bQ^* - (c + t) = 0 \\ \pi''(Q^*) &= -2b < 0.\end{aligned}$$

The critical point obtained from the first-order sufficient condition

$$Q^* = \frac{a - c - t}{2b}, \quad a - c - t > 0,$$

at the price

$$p^* = a - bQ^* = \frac{a + c + t}{2}$$

and the maximal profit is

$$\pi(Q^*) = \frac{(a - c - t)^2}{4b} - f.$$

We can perform the Comparative Static Analysis (Sensitivity Analysis). That is, how the optimal solution changes when a parameter changes,

$$\frac{dQ^*}{dt} = -\frac{1}{2b} < 0$$

$$\frac{dp^*}{dt} = \frac{1}{2}.$$

We also see how the objective function values changes when a parameter changes,

$$\frac{d\pi(Q^*; t)}{dt} = -\frac{a - c - t}{2b} < 0.$$

We will see in later chapters that the general theory for sensitivity analyses are based on the Implicit Function Theorem and Envelope Theorem.

HW What is the specific tax rate t that maximization tax revenue $T = tQ^* = t \frac{(a-c-t)}{2b}$?

HW Baldani, p. 62, #2.8. Redo all above when the tax is ad valorem $T = tpQ$.

3.5 Profit Maximization of Duopoly (Cournot Model) A market with two sellers face a common demand function

$$P = a - bQ,$$

where $Q = q_1 + q_2$, the sum of quantity sold by both firms. The profit functions of the two firms are given respectively by

$$\begin{aligned}\pi_1(q_1) &= TR_1 - TC_1 = aq_1 - bq_1q_2 - bq_1^2 - cq_1 \\ \pi_2(q_2) &= aq_2 - bq_1q_2 - bq_2^2 - cq_2\end{aligned}$$

Each firm is assumed to maximize its profit by assuming a given production quantity of the other's. The first-order necessary conditions product the so-called reaction functions as follows.

$$\begin{aligned}\pi'_1(q_1^*) &= a - bq_2 - 2bq_1^* - c = 0 \Rightarrow q_1^* = \frac{a - c}{2b} - \frac{1}{2}q_2 \\ \pi'_2(q_2^*) &= a - bq_1 - 2bq_2^* - c = 0 \Rightarrow q_2^* = \frac{a - c}{2b} - \frac{1}{2}q_1\end{aligned}$$

Solving the two linear reaction functions, we have the equilibrium

$$q_1^* = \frac{a - c}{3b} = q_2^*.$$

This is called a *Nash Equilibrium* because once it is in the equilibrium neither party sees incentive to change.

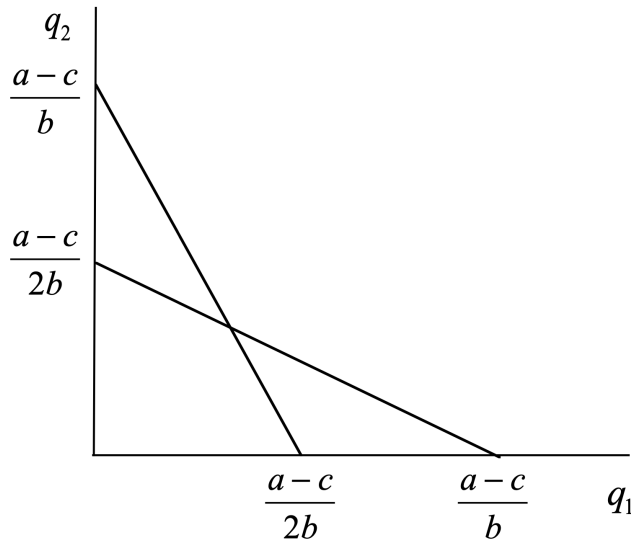


Figure 3.4 Reaction functions of two identical firms in a duopoly market.

The market quantity and price are given by

$$Q^* = q_1^* + q_2^* = 2 \left(\frac{a-c}{3b} \right)$$

$$P^* = a - bQ^* = \frac{a+2c}{3}.$$

HW Redo the Duopoly model with one of the following changes

- The two firms have different costs $c_1 \neq c_2$.
- Each firm has a fixed cost.

HW Baldani, p. 63, #2.11.

3.6 Balanced-Budget Multiplier Given a simple macroeconomic model as given below, when the government chooses to have a balance budget, the multiplier is equal to 1.

$$Y = C + I + G$$

$$C = a + bY_d, 0 < b < 1$$

$$Y_d = Y - T$$

$$I = I_0 - er, e > 0$$

By substitution, the equilibrium income is

$$Y = a + b(Y - T) + I_0 - er + G$$

$$Y^* = \frac{a - bT + I_0 - er + G}{1 - b}.$$

Taking derivative with respect to the government expenditures and tax, we have

$$\begin{aligned}\frac{\partial Y^*}{\partial G} &= \frac{1}{1 - b} > 0. \\ \frac{\partial Y^*}{\partial T} &= -\frac{b}{1 - b} < 0.\end{aligned}$$

The Balanced Budget Multiplier

$$\frac{\partial Y^*}{\partial G} + \frac{\partial Y^*}{\partial T} = \frac{1}{1 - b} - \frac{b}{1 - b} = 1.$$

We will see a more rigorous treatment in Chapter 5.

HW Baldani, p.64, #2.15.