

# Expected Utility and Risk Aversion

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# Introduction

Expected utility is the standard framework for modeling investor choices. The following topics will be covered:

- 1 Analyze conditions on individual preferences that lead to an expected utility function.
- 2 Consider the link between utility, risk aversion, and risk premia for particular assets.
- 3 Examine how risk aversion affects an individual's portfolio choice between a risky and riskfree asset.

## Preferences when Returns are Uncertain

- Economists typically analyze the price of a good using supply and demand. We can do the same for assets.
- The main distinction between assets is their future payoffs: Risky assets have uncertain payoffs, so a theory of asset demands must specify investor preferences over different, uncertain payoffs.
- Consider relevant criteria for ranking preferences. One possible measure is the asset's average payoff.

## Criterion: Expected Payoff

- Suppose an asset offers a single random payoff at a particular future date, and this payoff has a discrete distribution with  $n$  possible outcomes  $(x_1, \dots, x_n)$  and corresponding probabilities  $(p_1, \dots, p_n)$ , where  $\sum_{i=1}^n p_i = 1$  and  $p_i \geq 0$ .
- Then the expected value of the payoff (or, more simply, the expected payoff) is  $\bar{x} \equiv E[\tilde{x}] = \sum_{i=1}^n p_i x_i$ .
- Is an asset's expected value a suitable criterion for determining an individual's demand for the asset?
- Consider how much Paul would pay Peter to play the following coin flipping game.

## St. Petersburg Paradox, Nicholas Bernoulli, 1713

- Peter continues to toss a coin until it lands “heads.” He agrees to give Paul one ducat if he gets heads on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on.
- If the number of coin flips taken to first obtain heads is  $i$ , then  $p_i = \left(\frac{1}{2}\right)^i$  and  $x_i = 2^{i-1}$ . Thus, Paul's expected payoff equals

$$\begin{aligned}\bar{x} &= \sum_{i=1}^{\infty} p_i x_i = \frac{1}{2}1 + \frac{1}{4}2 + \frac{1}{8}4 + \frac{1}{16}8 + \dots & (1) \\ &= \frac{1}{2}(1 + \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + \dots \\ &= \frac{1}{2}(1 + 1 + 1 + 1 + \dots = \infty\end{aligned}$$

# St. Petersburg Paradox

- What is the paradox?
- Daniel Bernoulli (1738) explained it using *expected utility*.
- His insight was that an individual's utility from receiving a payoff differed from the size of the payoff.
- Instead of valuing an asset as  $\bar{x} = \sum_{i=1}^n p_i x_i$ , its value,  $V$ , would be

$$V \equiv E[U(\tilde{x})] = \sum_{i=1}^n p_i U_i$$

where  $U_i$  is the utility associated with payoff  $x_i$ .

- He hypothesized that  $U_i$  is diminishingly increasing in wealth.

## Criterion: Expected Utility

- Von Neumann and Morgenstern (1944) derived conditions on an individual's preferences that, if satisfied, would make them consistent with an expected utility function.
- Define a *lottery* as an asset that has a risky payoff and consider an individual's optimal choice of a lottery from a given set of different lotteries. The possible payoffs of all lotteries are contained in the set  $\{x_1, \dots, x_n\}$ .
- A lottery is characterized by an ordered set of probabilities  $P = \{p_1, \dots, p_n\}$ , where of course,  $\sum_{i=1}^n p_i = 1$  and  $p_i \geq 0$ . Let a different lottery be  $P^* = \{p_1^*, \dots, p_n^*\}$ . Let  $\succ$ ,  $\prec$ , and  $\sim$  denote preference and indifference between lotteries.

# Preferences Over Different Random Payoffs

- Specifically, if an individual prefers lottery  $P^*$  to lottery  $P$ , this can be denoted as  $P^* \succ P$  or  $P \prec P^*$ .
- When the individual is indifferent between the two lotteries, this is written as  $P^* \sim P$ .
- If an individual prefers lottery  $P^*$  to lottery  $P$  or she is indifferent between lotteries  $P^*$  and  $P$ , this is written as  $P^* \succeq P$  or  $P \preceq P^*$ .
- N.B.: all lotteries have the same payoff set  $\{x_1, \dots, x_n\}$ , so we focus on the (different) probability sets  $P$  and  $P^*$ .

## Expected Utility Axioms 1-3

- *Theorem:* There exists an expected utility function  $V(p_1, \dots, p_n)$  if the following axioms hold:

### Axioms:

#### 1) *Completeness*

For any two lotteries  $P^*$  and  $P$ , either  $P^* \succ P$ , or  $P^* \prec P$ , or  $P^* \sim P$ .

#### 2) *Transitivity*

If  $P^{**} \succeq P^*$  and  $P^* \succeq P$ , then  $P^{**} \succeq P$ .

#### 3) *Continuity*

If  $P^{**} \succeq P^* \succeq P$ , there exists some  $\lambda \in [0, 1]$  such that  $P^* \sim \lambda P^{**} + (1 - \lambda)P$ , where  $\lambda P^{**} + (1 - \lambda)P$  denotes a “compound lottery”; namely, with probability  $\lambda$  one receives the lottery  $P^{**}$  and with probability  $(1 - \lambda)$  one receives the lottery  $P$ .

## Expected Utility Axioms 4-5

### 4) *Independence*

For any two lotteries  $P$  and  $P^*$ ,  $P^* \succ P$  if and only if for all  $\lambda \in (0,1]$  and all  $P^{**}$ :

$$\lambda P^* + (1 - \lambda)P^{**} \succ \lambda P + (1 - \lambda)P^{**}$$

Moreover, for any two lotteries  $P$  and  $P^\dagger$ ,  $P \sim P^\dagger$  if and only if for all  $\lambda \in (0,1]$  and all  $P^{**}$ :

$$\lambda P + (1 - \lambda)P^{**} \sim \lambda P^\dagger + (1 - \lambda)P^{**}$$

### 5) *Dominance*

Let  $P^1$  be the compound lottery  $\lambda_1 P^\ddagger + (1 - \lambda_1)P^\dagger$  and  $P^2$  be the compound lottery  $\lambda_2 P^\ddagger + (1 - \lambda_2)P^\dagger$ . If  $P^\ddagger \succ P^\dagger$ , then  $P^1 \succ P^2$  if and only if  $\lambda_1 > \lambda_2$ .

## Discussion: Machina (1987)

- The first three axioms are analogous to those used to establish a real-valued utility function in consumer choice theory.
- Axiom 4 (Independence) is novel, but its linearity property is critical for preferences to be consistent with expected utility.
- To understand its meaning, suppose an individual chooses  $P^* \succ P$ . By Axiom 4, the choice between  $\lambda P^* + (1 - \lambda)P^{**}$  and  $\lambda P + (1 - \lambda)P^{**}$  is equivalent to tossing a coin that with probability  $(1 - \lambda)$  lands “tails,” in which both lotteries pay  $P^{**}$ , and with probability  $\lambda$  lands “heads,” in which case the individual should prefer  $P^*$  to  $P$ .

# Allais Paradox

- But, there is some experimental evidence counter to this axiom.
- Consider lotteries over  $\{x_1, x_2, x_3\} = \{\$0, \$1m, \$5m\}$  and two lottery choices:  
C1:  $P^1 = \{0, 1, 0\}$  vs  $P^2 = \{.01, .89, .1\}$   
C2:  $P^3 = \{.9, 0, .1\}$  vs  $P^4 = \{.89, .11, 0\}$
- Which do you choose in C1? In C2?

# Allais Paradox

- Experimental evidence suggests most people prefer  $P^1 \succ P^2$  and  $P^3 \succ P^4$ .
- But this violates Axiom 4. Why?
- Define  $P^5 = \{1/11, 0, 10/11\}$  and let  $\lambda = 0.11$ . Note that  $P^2$  is equivalent to the compound lottery:

$$\begin{aligned}P^2 &\sim \lambda P^5 + (1 - \lambda) P^1 \\ &\sim 0.11\{1/11, 0, 10/11\} + 0.89\{0, 1, 0\} \\ &\sim \{.01, .89, .1\}\end{aligned}$$

# Allais Paradox

- Note also that  $P^1$  is trivially the compound lottery  $\lambda P^1 + (1 - \lambda) P^1$ . Hence, if  $P^1 \succ P^2$ , the independence axiom implies  $P^1 \succ P^5$ .
- Now also define  $P^6 = \{1, 0, 0\}$ , and note that  $P^3$  equals the following compound lottery:

$$\begin{aligned} P^3 &\sim \lambda P^5 + (1 - \lambda) P^6 \\ &\sim 0.11\{1/11, 0, 10/11\} + 0.89\{1, 0, 0\} \\ &\sim \{.9, 0, .1\} \end{aligned}$$

while  $P^4$  is equivalent to the compound lottery

$$\begin{aligned} P^4 &\sim \lambda P^1 + (1 - \lambda) P^6 \\ &\sim 0.11\{0, 1, 0\} + 0.89\{1, 0, 0\} \\ &\sim \{.89, 0.11, 0\} \end{aligned}$$

# Allais Paradox

- But if  $P^3 \succ P^4$ , the independence axiom implies  $P^5 \succ P^1$ , which contradicts the choice of  $P^1 \succ P^2$  that implies  $P^1 \succ P^5$ .
- Despite the sometimes contradictory experimental evidence, expected utility is still the dominant paradigm.
- However, we will consider different models of utility at a later date, including those that reflect psychological biases.

## Deriving Expected Utility: Axiom 1

- We now prove the theorem by showing that if an individual's preferences over lotteries satisfy the preceding axioms, these preferences can be ranked by the individual's expected utility of the lotteries.
- Define an "elementary" or "primitive" lottery,  $e_i$ , which returns outcome  $x_i$  with probability 1 and all other outcomes with probability zero, that is,  $e_i = \{p_1, \dots, p_{i-1}, p_i, p_{i+1}, \dots, p_n\} = \{0, \dots, 0, 1, 0, \dots, 0\}$  where  $p_i = 1$  and  $p_j = 0 \forall j \neq i$ .
- Without loss of generality, assume that the outcomes are ordered such that  $e_n \succeq e_{n-1} \succeq \dots \succeq e_1$ . This follows from the completeness axiom for this case of  $n$  elementary lotteries

## Deriving Expected Utility: Axiom 3, Axiom 4

- From the continuity axiom, for each  $e_i$ , there exists a  $U_i \in [0, 1]$  such that

$$e_i \sim U_i e_n + (1 - U_i) e_1 \quad (2)$$

and for  $i = 1$ , this implies  $U_1 = 0$  and for  $i = n$ , this implies  $U_n = 1$ .

- Now a given arbitrary lottery,  $P = \{p_1, \dots, p_n\}$ , can be viewed as a compound lottery over the  $n$  elementary lotteries, where elementary lottery  $e_i$  is obtained with probability  $p_i$ .

$$P \sim p_1 e_1 + \dots + p_n e_n$$

## Deriving Expected Utility: Axiom 4

- By the independence axiom, and equation (2), the individual is indifferent between lottery,  $P$ , and the following lottery:

$$p_1 e_1 + \dots + p_n e_n \sim p_1 e_1 + \dots + p_{i-1} e_{i-1} + p_i [U_i e_n + (1 - U_i) e_1] + p_{i+1} e_{i+1} + \dots + p_n e_n \quad (3)$$

where the indifference relation in equation (2) substitutes for  $e_i$  on the right-hand side of (3).

- By repeating this substitution for all  $i$ ,  $i = 1, \dots, n$ , the individual will be indifferent between  $P$  and

$$p_1 e_1 + \dots + p_n e_n \sim \left( \sum_{i=1}^n p_i U_i \right) e_n + \left( 1 - \sum_{i=1}^n p_i U_i \right) e_1 \quad (4)$$

## Deriving Expected Utility: Axiom 5

- Now define  $\Lambda \equiv \sum_{i=1}^n p_i U_i$ . Thus,  $P \sim \Lambda e_n + (1 - \Lambda)e_1$
- Similarly, we can show that any other arbitrary lottery  $P^* = \{p_1^*, \dots, p_n^*\} \sim \Lambda^* e_n + (1 - \Lambda^*)e_1$ , where  $\Lambda^* \equiv \sum_{i=1}^n p_i^* U_i$ .
- We know from the dominance axiom that  $P^* \succ P$  iff  $\Lambda^* > \Lambda$ , implying  $\sum_{i=1}^n p_i^* U_i > \sum_{i=1}^n p_i U_i$ .
- So we can define the function

$$V(p_1, \dots, p_n) = \sum_{i=1}^n p_i U_i \quad (5)$$

which implies that  $P^* \succ P$  iff  $V(p_1^*, \dots, p_n^*) > V(p_1, \dots, p_n)$ .

## Deriving Expected Utility: The End

- The function in (5) is known as von Neumann-Morgenstern expected utility. It is linear in the probabilities and is unique up to a linear monotonic transformation.
- The intuition for why expected utility is unique up to a linear transformation comes from equation (2). Here we express elementary lottery  $i$  in terms of the least and most preferred elementary lotteries. However, other bases for ranking a given lottery are possible.
- For  $U_i = U(x_i)$ , an individual's choice over lotteries is the same under the transformation  $aU(x_i) + b$ , but not a nonlinear transformation that changes the “shape” of  $U(x_i)$ .

## St. Petersburg Paradox Revisited

- Suppose  $U_i = U(x_i) = \sqrt{x_i}$ . Then the expected utility of the St. Petersburg payoff is

$$\begin{aligned}
 V &= \sum_{i=1}^n p_i U_i = \sum_{i=1}^{\infty} \frac{1}{2^i} \sqrt{2^{i-1}} = \sum_{i=1}^{\infty} 2^{-\frac{1}{2}(i+1)} = \sum_{i=2}^{\infty} 2^{-\frac{i}{2}} \\
 &= 2^{-\frac{2}{2}} + 2^{-\frac{3}{2}} + \dots \\
 &= \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^i - 1 - \frac{1}{\sqrt{2}} = \frac{1}{1 - \frac{1}{\sqrt{2}}} - 1 - \frac{1}{\sqrt{2}} \\
 &= \frac{1}{2 - \sqrt{2}} \cong 1.707
 \end{aligned}$$

A certain payment of  $1.707^2 \cong 2.914$  ducats has the same expected utility as playing the St. Petersburg game.

## Super St. Petersburg

- The St. Petersburg game has infinite expected payoff because the probability of winning declines at rate  $2^i$ , while the winning payoff increases at rate  $2^i$ .
- In a “super” St. Petersburg paradox, we can make the winning payoff increase at a rate  $x_i = U^{-1}(2^{i-1})$  to cause expected utility to increase at  $2^i$ . For square-root utility,  $x_i = (2^i 2)^2 = 2^{2i-2}$ ; that is,  $x_1 = 1$ ,  $x_2 = 4$ ,  $x_3 = 16$ , and so on. The expected utility of “super” St. Petersburg is

$$V = \sum_{i=1}^n p_i U_i = \sum_{i=1}^{\infty} \frac{1}{2^i} \sqrt{2^{2i-2}} = \sum_{i=1}^{\infty} \frac{1}{2^i} 2^{i-1} = \infty \quad (6)$$

- Should we be concerned that if prizes grow quickly enough, we can get infinite expected utility (and valuations) for any chosen form of expected utility function?

## Von Neumann-Morgenstern Utility

- The von Neumann-Morgenstern expected utility can be generalized to a continuum of outcomes and lotteries with continuous probability distributions. Analogous to equation (5) is

$$V(F) = E[U(\tilde{x})] = \int U(x) dF(x) = \int U(x) f(x) dx \quad (7)$$

where  $F(x)$  is the lottery's cumulative distribution function over the payoffs,  $x$ .  $V$  can be written in terms of the probability density,  $f(x)$ , when  $F(x)$  is absolutely continuous.

- This is analogous to our previous lottery represented by the discrete probabilities  $P = \{p_1, \dots, p_n\}$ .

# Risk Aversion

- Diminishing marginal utility results in risk aversion: being unwilling to accept a “fair” lottery. Why?
- Let there be a lottery that has a random payoff,  $\tilde{\varepsilon}$ , where

$$\tilde{\varepsilon} = \begin{cases} \varepsilon_1 \text{ with probability } p \\ \varepsilon_2 \text{ with probability } 1 - p \end{cases} \quad (8)$$

- The requirement that it be a “fair” lottery restricts its expected value to equal zero:

$$E[\tilde{\varepsilon}] = p\varepsilon_1 + (1 - p)\varepsilon_2 = 0 \quad (9)$$

which implies  $\varepsilon_1/\varepsilon_2 = -(1 - p)/p$ , or solving for  $p$ ,  $p = -\varepsilon_2/(\varepsilon_1 - \varepsilon_2)$ . Since  $0 < p < 1$ ,  $\varepsilon_1$  and  $\varepsilon_2$  are of opposite signs.

## Risk Aversion and Concave Utility

- Suppose a vN-M maximizer with current wealth  $W$  is offered a fair lottery. Would he accept it?
- With the lottery, expected utility is  $E[U(W + \tilde{\varepsilon})]$ . Without it, expected utility is  $E[U(W)] = U(W)$ . Rejecting it implies

$$U(W) > E[U(W + \tilde{\varepsilon})] = pU(W + \varepsilon_1) + (1 - p)U(W + \varepsilon_2) \quad (10)$$

- $U(W)$  can be written as

$$U(W) = U(W + p\varepsilon_1 + (1 - p)\varepsilon_2) \quad (11)$$

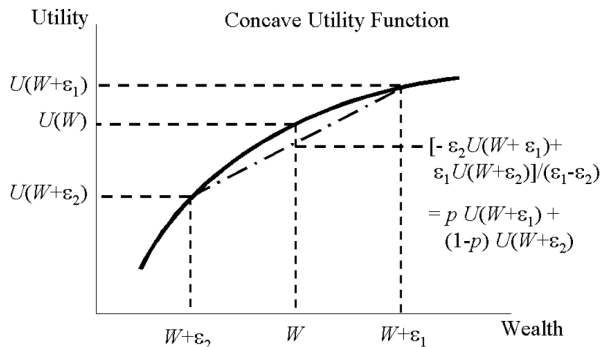
- Substituting into (10), we have

$$U(W + p\varepsilon_1 + (1 - p)\varepsilon_2) > pU(W + \varepsilon_1) + (1 - p)U(W + \varepsilon_2) \quad (12)$$

which is the definition of  $U$  being a concave function.

# Risk Aversion $\Leftrightarrow$ Concavity

- A function is concave if a line joining any two points lies entirely below the function. When  $U(W)$  is a continuous, second differentiable function, concavity implies  $U''(W) < 0$ .



## Risk Aversion $\Leftrightarrow$ Concavity

- To show that concave utility implies rejecting a fair lottery, we can use Jensen's inequality which says that for concave  $U(\cdot)$

$$E[U(\tilde{x})] < U(E[\tilde{x}]) \quad (13)$$

- Therefore, substituting  $\tilde{x} = W + \tilde{\varepsilon}$  with  $E[\tilde{\varepsilon}] = 0$ , we have

$$E[U(W + \tilde{\varepsilon})] < U(E[W + \tilde{\varepsilon}]) = U(W) \quad (14)$$

which is the desired result.

## Risk Aversion and Risk Premium

- How might aversion to risk be quantified? One way is to define a *risk premium* as the amount that an individual is willing to pay to avoid a risk.
- Let  $\pi$  denote the individual's risk premium for a lottery,  $\tilde{\varepsilon}$ .  $\pi$  is the maximum insurance payment an individual would pay to avoid the lottery risk:

$$U(W - \pi) = E[U(W + \tilde{\varepsilon})] \quad (15)$$

- $W - \pi$  is defined as the *certainty equivalent* level of wealth associated with the lottery,  $\tilde{\varepsilon}$ .
- For concave utility, Jensen's inequality implies  $\pi > 0$  when  $\tilde{\varepsilon}$  is fair: the individual would accept wealth lower than her expected wealth following the lottery,  $E[W + \tilde{\varepsilon}]$ , to avoid the lottery.

# Risk Premium

- For small  $\tilde{\varepsilon}$  we can take a Taylor approximation of equation (15) around  $\tilde{\varepsilon} = 0$  and  $\pi = 0$ .
- Expanding the left-hand side about  $\pi = 0$  gives

$$U(W - \pi) \cong U(W) - \pi U'(W) \quad (16)$$

and expanding the right-hand side about  $\tilde{\varepsilon}$  gives

$$\begin{aligned} E[U(W + \tilde{\varepsilon})] &\cong E\left[U(W) + \tilde{\varepsilon}U'(W) + \frac{1}{2}\tilde{\varepsilon}^2U''(W)\right] \quad (17) \\ &= U(W) + 0 + \frac{1}{2}\sigma^2U''(W) \end{aligned}$$

where  $\sigma^2 \equiv E[\tilde{\varepsilon}^2]$  is the lottery's variance.

## Risk Premium cont'd

- Equating the results in (16) and (17) gives

$$\pi = -\frac{1}{2}\sigma^2 \frac{U''(W)}{U'(W)} \equiv \frac{1}{2}\sigma^2 R(W) \quad (18)$$

where  $R(W) \equiv -U''(W)/U'(W)$  is the Pratt (1964)-Arrow (1971) measure of absolute risk aversion.

- Since  $\sigma^2 > 0$ ,  $U'(W) > 0$ , and  $U''(W) < 0$ , concavity of the utility function ensures that  $\pi$  must be positive
- An individual may be very risk averse ( $-U''(W)$  is large), but may be unwilling to pay a large risk premium if he is poor since his marginal utility  $U'(W)$  is high.

## $-U''(W)$ and $U'(W)$

- Consider the following *negative exponential* utility function:

$$U(W) = -e^{-bW}, b > 0 \quad (19)$$

- Note that  $U'(W) = be^{-bW} > 0$  and  $U''(W) = -b^2e^{-bW} < 0$ .
- Consider the behavior of a very wealthy individual whose wealth approaches infinity

$$\lim_{W \rightarrow \infty} U'(W) = \lim_{W \rightarrow \infty} U''(W) = 0 \quad (20)$$

- There's no concavity, so is there no risk aversion?

$$R(W) = \frac{b^2e^{-bW}}{be^{-bW}} = b \quad (21)$$

## Absolute Risk Aversion: Dollar Payment for Risk

- We see that negative exponential utility,  $U(W) = -e^{-bW}$ , has constant absolute risk aversion.
- If, instead, we want absolute risk aversion to decline in wealth, a necessary condition is that the utility function must have a positive third derivative:

$$\frac{\partial R(W)}{\partial W} = \frac{\partial -\frac{U''(W)}{U'(W)}}{\partial W} = -\frac{U'''(W)U'(W) - [U''(W)]^2}{[U'(W)]^2} \quad (22)$$

$R(W) \Rightarrow U(W)$ 

- The coefficient of risk aversion contains all relevant information about the individual's risk preferences. Note that

$$R(W) = -\frac{U''(W)}{U'(W)} = -\frac{\partial (\ln [U'(W)])}{\partial W} \quad (23)$$

- Integrating both sides of (23), we have

$$-\int R(W)dW = \ln[U'(W)] + c_1 \quad (24)$$

where  $c_1$  is an arbitrary constant. Taking the exponential function of (24) gives

$$e^{-\int R(W)dW} = U'(W)e^{c_1} \quad (25)$$

# $R(W) \Rightarrow U(W)$ cont'd

- Integrating once again, we obtain

$$\int e^{-\int R(W)dW} dW = e^{c_1} U(W) + c_2 \quad (26)$$

where  $c_2$  is another arbitrary constant.

- Because vN-M expected utility functions are unique up to a linear transformation,  $e^{c_1} U(W) + c_2$  reflects the same risk preferences as  $U(W)$ .

## Relative Risk Aversion

- Relative risk aversion is another frequently used measure defined as

$$R_r(W) = WR(W) \quad (27)$$

- Consider risk aversion for some utility functions often used in models of portfolio choice and asset pricing. *Power* utility can be written as

$$U(W) = \frac{1}{\gamma} W^\gamma, \gamma < 1 \quad (28)$$

implying that  $R(W) = -\frac{(\gamma-1)W^{\gamma-2}}{W^{\gamma-1}} = \frac{(1-\gamma)}{W}$  and, therefore,  $R_r(W) = 1 - \gamma$ .

- Hence, it displays *constant relative risk aversion*.

## Logarithmic Utility: Constant Relative Risk Aversion

- Logarithmic utility is a limiting case of power utility. Since utility functions are unique up to a linear transformation, write the power utility function as

$$\frac{1}{\gamma} W^\gamma - \frac{1}{\gamma} = \frac{W^\gamma - 1}{\gamma}$$

- Next take its limit as  $\gamma \rightarrow 0$ . Do so by rewriting the numerator and applying L'Hôpital's rule:

$$\lim_{\gamma \rightarrow 0} \frac{W^\gamma - 1}{\gamma} = \lim_{\gamma \rightarrow 0} \frac{e^{\gamma \ln(W)} - 1}{\gamma} = \lim_{\gamma \rightarrow 0} \frac{\ln(W) W^\gamma}{1} = \ln(W) \quad (29)$$

- Thus, logarithmic utility is power utility with coefficient of relative risk aversion  $(1 - \gamma) = 1$  since  $R(W) = -\frac{W^{-2}}{W^{-1}} = -\frac{1}{W}$  and  $R_r(W) = 1$ .

## HARA: Power, Log, Quadratic

- Hyperbolic absolute-risk-aversion (HARA) utility generalizes all of the previous utility functions:

$$U(W) = \frac{1-\gamma}{\gamma} \left( \frac{\alpha W}{1-\gamma} + \beta \right)^\gamma \quad (30)$$

*s.t.*  $\gamma \neq 1$ ,  $\alpha > 0$ ,  $\frac{\alpha W}{1-\gamma} + \beta > 0$ , and  $\beta = 1$  if  $\gamma = -\infty$ .

- Thus,  $R(W) = \left( \frac{W}{1-\gamma} + \frac{\beta}{\alpha} \right)^{-1}$ . Since  $R(W)$  must be  $> 0$ , it implies  $\beta > 0$  when  $\gamma > 1$ .  $R_r(W) = W \left( \frac{W}{1-\gamma} + \frac{\beta}{\alpha} \right)^{-1}$ .
- HARA utility nests constant absolute risk aversion ( $\gamma = -\infty$ ,  $\beta = 1$ ), constant relative risk aversion ( $\gamma < 1$ ,  $\beta = 0$ ), and quadratic ( $\gamma = 2$ ) utility functions.

## Another Look at the Risk Premium

- A premium to avoid risk is fine for insurance, but we may also be interested in a premium to *bear* risk.
- This alternative concept of a risk premium was used by Arrow (1971), identical to the earlier one by Pratt (1964).
- Suppose that a fair lottery  $\tilde{\varepsilon}$ , has the following payoffs and probabilities:

$$\tilde{\varepsilon} = \begin{cases} +\epsilon & \text{with probability } \frac{1}{2} \\ -\epsilon & \text{with probability } \frac{1}{2} \end{cases} \quad (31)$$

- How much do we need to deviate from “fairness” to make a risk-averse individual indifferent to this lottery?

## Risk Premium v2

- Let's define a risk premium,  $\theta$ , in terms of probability of winning  $p$ :

$$\theta = \text{Prob}(\text{win}) - \text{Prob}(\text{lose}) = p - (1 - p) = 2p - 1 \quad (32)$$

- Therefore, from (32) we have

$$\begin{aligned}\text{Prob}(\text{win}) &\equiv p = \frac{1}{2}(1 + \theta) \\ \text{Prob}(\text{lose}) &= 1 - p = \frac{1}{2}(1 - \theta)\end{aligned}$$

- We want  $\theta$  that equalizes the utilities of taking and not taking the lottery:

$$U(W) = \frac{1}{2}(1 + \theta)U(W + \epsilon) + \frac{1}{2}(1 - \theta)U(W - \epsilon) \quad (33)$$

## Risk Aversion (again)

- Let's again take a Taylor approximation of the right side, around  $\epsilon = 0$

$$\begin{aligned}U(W) &= \frac{1}{2}(1 + \theta) [U(W) + \epsilon U'(W) + \frac{1}{2}\epsilon^2 U''(W)] \quad (34) \\ &\quad + \frac{1}{2}(1 - \theta) [U(W) - \epsilon U'(W) + \frac{1}{2}\epsilon^2 U''(W)] \\ &= U(W) + \epsilon\theta U'(W) + \frac{1}{2}\epsilon^2 U''(W)\end{aligned}$$

- Rearranging (34) implies

$$\theta = \frac{1}{2}\epsilon R(W) \quad (35)$$

which, as before, is a function of the coefficient of absolute risk aversion.

## Risk Aversion (again)

- Note that the Arrow premium,  $\theta$ , is in terms of a probability, while the Pratt measure,  $\pi$ , is in units of a monetary payment.
- If we multiply  $\theta$  by the monetary payment received,  $\epsilon$ , then equation (35) becomes

$$\epsilon\theta = \frac{1}{2}\epsilon^2 R(W) \quad (36)$$

- Since  $\epsilon^2$  is the variance of the random payoff,  $\tilde{\epsilon}$ , equation (36) shows that the Pratt and Arrow risk premia are equivalent. Both were obtained as a linearization of the true function around  $\tilde{\epsilon} = 0$ .

## A Simple Portfolio Choice Problem

- Let's consider an individual's single-period portfolio choice problem.
- Assume there is a riskless security that pays a rate of return equal to  $r_f$  and just one risky security that pays a random rate of return equal to  $\tilde{r}$ .
- Also, let  $W_0$  be the individual's initial wealth, and let  $A$  be the monetary amount that the individual invests in the risky asset at the beginning of the period. Thus,  $W_0 - A$  is the initial investment in the riskless security.
- The individual's end-of-period wealth,  $\tilde{W}$ , is given by:

$$\begin{aligned}\tilde{W} &= (W_0 - A)(1 + r_f) + A(1 + \tilde{r}) \\ &= W_0(1 + r_f) + A(\tilde{r} - r_f)\end{aligned}\tag{37}$$

## Single Period Utility Maximization

- The individual chooses  $A$  to maximize expected utility that is increasing and concave in end-of-period wealth:

$$\max_A E[U(\tilde{W})] = \max_A E[U(W_0(1+r_f) + A(\tilde{r} - r_f))] \quad (38)$$

- The first-order condition with respect to  $A$  is:

$$E[U'(\tilde{W})(\tilde{r} - r_f)] = 0 \quad (39)$$

- Note that the second order condition

$$E[U''(\tilde{W})(\tilde{r} - r_f)^2] \leq 0 \quad (40)$$

is satisfied because  $U''(\tilde{W}) \leq 0$  from concavity.

## Obtaining $A^*$ from FOC

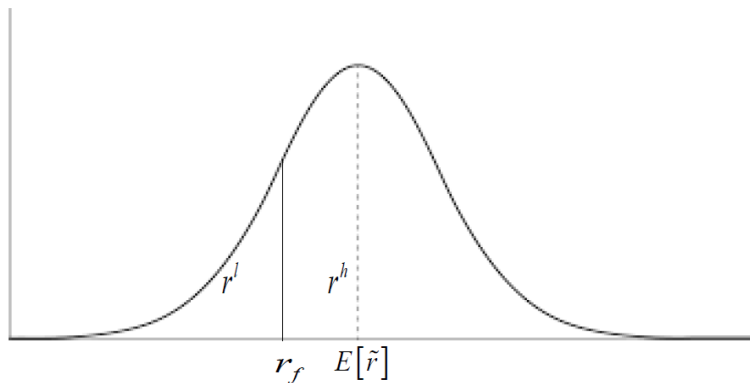
- Suppose  $E[\tilde{r}] = r_f$ . Then we can show  $A=0$  is the solution.
- If  $A=0$ , then  $\tilde{W} = W_0(1 + r_f)$  so that  $U'(\tilde{W}) = U'(W_0(1 + r_f))$  is nonstochastic. Hence,  $E[U'(\tilde{W})(\tilde{r} - r_f)] = U'(W_0(1 + r_f))E[\tilde{r} - r_f] = 0$ .
- Next, suppose  $E[\tilde{r}] > r_f$ .
- $A = 0$  is not a solution because  $E[U'(\tilde{W})(\tilde{r} - r_f)] = U'(W_0(1 + r_f))E[\tilde{r} - r_f] > 0$  when  $A = 0$ .
- Thus, when  $E[\tilde{r}] - r_f > 0$ , let's show that  $A > 0$ .

## Why must $A > 0$ ?

- Let  $r^h$  denote a realization of  $\tilde{r} > r_f$ , and let  $W^h$  be the corresponding level of  $\tilde{W}$
- Also, let  $r^l$  denote a realization of  $\tilde{r} < r_f$ , and let  $W^l$  be the corresponding level of  $\tilde{W}$ .
- Then  $U'(W^h)(r^h - r_f) > 0$  and  $U'(W^l)(r^l - r_f) < 0$ .
- For  $U'(\tilde{W})(\tilde{r} - r_f)$  to average to zero for all realizations of  $\tilde{r}$ , it must be that  $W^h > W^l$  so that  $U'(W^h) < U'(W^l)$  due to the concavity of the utility function.
- Why? Since  $E[\tilde{r}] > r_f$ , the average  $r^h$  is farther above  $r_f$  than the average  $r^l$  is below  $r_f$ . To preserve (39), the multipliers must satisfy  $U'(W^h) < U'(W^l)$  to compensate, which occurs when  $W^h > W^l$  and which requires that  $A > 0$ .

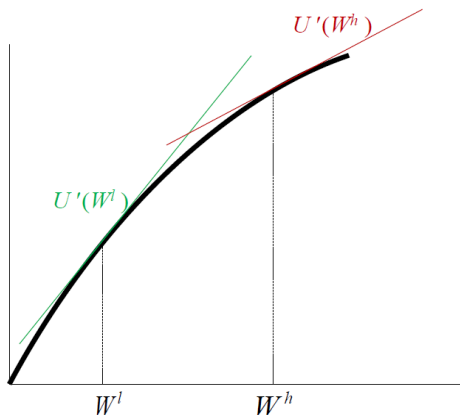
# Distribution of Returns

- When  $E[\tilde{r}] > r_f$ , there is more probability mass for  $r^h$  than  $r^l$ .



# Concave Utility

- When  $E[\tilde{r}] > r_f$ , need  $U'(W^h) < U'(W^l)$  for  $E[U'(\tilde{W})(\tilde{r} - r_f)] = 0$ .



## How does optimal $A$ change with initial wealth, $W_0$ ?

- Let us use implicit differentiation to obtain  $\frac{dA(W_0)}{dW_0}$ .
- Define  $f(A, W_0) \equiv E \left[ U \left( \widetilde{W} \right) \right]$ , and let maximized expected utility when  $A$  is optimally chosen be  $v(W_0) = \max_A f(A, W_0)$ .
- Also define  $A(W_0)$  as the value of  $A$  that maximizes  $f$  for a given initial wealth,  $W_0$ .
- Using the chain rule, the total derivative of  $v(W_0)$  with respect to  $W_0$  is  $\frac{dv(W_0)}{dW_0} = \frac{\partial f(A, W_0)}{\partial A} \frac{dA(W_0)}{dW_0} + \frac{\partial f(A(W_0), W_0)}{\partial W_0}$ .
- $\frac{\partial f(A, W_0)}{\partial A} = 0$  since it is the first-order condition for a maximum.

## How does $A$ change wrt $W_0$ cont'd

- The total derivative simplifies to  $\frac{dv(W_0)}{dW_0} = \frac{\partial f(A(W_0), W_0)}{\partial W_0}$ , implying that the maximized objective function with respect to a parameter is just the partial derivative with respect to that parameter.
- Second, consider how the optimal value of the control variable,  $A(W_0)$ , changes when the parameter  $W_0$  changes.
- We do so by taking the total derivative of the F.O.C. (39),  $\partial f(A(W_0), W_0) / \partial A = 0$ , with respect to  $W_0$ :

$$\frac{\partial(\partial f(A(W_0), W_0) / \partial A)}{\partial W_0} = 0 = \frac{\partial^2 f(A(W_0), W_0)}{\partial A^2} \frac{dA(W_0)}{dW_0} + \frac{\partial^2 f(A(W_0), W_0)}{\partial A \partial W_0}$$

## How does $A$ change wrt $W_0$ cont'd

- Rearranging the above gives us

$$\frac{dA(W_0)}{dW_0} = -\frac{\partial^2 f(A(W_0), W_0)}{\partial A \partial W_0} / \frac{\partial^2 f(A(W_0), W_0)}{\partial A^2} \quad (41)$$

- We can then evaluate it to obtain

$$\frac{dA}{dW_0} = \frac{(1 + r_f)E[U''(\tilde{W})(\tilde{r} - r_f)]}{-E[U''(\tilde{W})(\tilde{r} - r_f)^2]} \quad (42)$$

- The denominator of (42) is positive because of concavity, so the sign of  $\frac{dA}{dW_0}$  depends on the numerator.

## Implications for $\frac{dA}{dW_0}$ with DARA

- Consider an individual with absolute risk aversion that is decreasing in wealth. Assuming  $E[\tilde{r}] > r_f$  so that  $A > 0$ :

$$R(W^h) < R(W_0(1 + r_f)) \quad (43)$$

where, as before,  $R(W) = -U''(W)/U'(W)$ .

- Multiplying both sides of (43) by  $-U'(W^h)(r^h - r_f)$ , which is a negative quantity, the inequality sign reverses:

$$U''(W^h)(r^h - r_f) > -U'(W^h)(r^h - r_f)R(W_0(1 + r_f)) \quad (44)$$

- Then for  $A > 0$ , we have  $W^l < W_0(1 + r_f)$ . If absolute risk aversion is decreasing in wealth, this implies

$$R(W^l) > R(W_0(1 + r_f)) \quad (45)$$

## Implications for $\frac{dA}{dW_0}$ with DARA

- Multiplying (45) by  $-U'(W^l)(r^l - r_f)$ , which is positive, inequality (45) becomes

$$U''(W^l)(r^l - r_f) > -U'(W^l)(r^l - r_f)R(W_0(1 + r_f)) \quad (46)$$

- Inequalities (44) and (46) are the same whether the realization is  $\tilde{r} = r^h$  or  $\tilde{r} = r^l$ , so taking expectations over all realizations of  $\tilde{r}$  implies

$$E \left[ U''(\tilde{W})(\tilde{r} - r_f) \right] > -E \left[ U'(\tilde{W})(\tilde{r} - r_f) \right] R(W_0(1 + r_f)) \quad (47)$$

- The first term on the right-hand side is just the FOC.

## Implications for risk-taking with ARA/RRA

- Therefore, inequality (47) reduces to

$$E \left[ U''(\tilde{W})(\tilde{r} - r_f) \right] > 0 \quad (48)$$

- Thus, DARA  $\Rightarrow dA/dW_0 > 0$ :  $A$  increases with initial wealth.
- What about the *proportion* of initial wealth? To analyze this, define

$$\eta \equiv \frac{\frac{dA}{dW_0}}{\frac{A}{W_0}} = \frac{dA}{dW_0} \frac{W_0}{A} \quad (49)$$

which is the elasticity measuring the proportional increase in the risky asset for an increase in initial wealth.

## Implications for risk-taking with RRA

- Adding  $1 - \frac{A}{A}$  to the right-hand side of (49) gives

$$\eta = 1 + \frac{(dA/dW_0)W_0 - A}{A} \quad (50)$$

- Substituting  $dA/dW_0$  from equation (42), we have

$$\eta = 1 + \frac{W_0(1 + r_f)E \left[ U''(\tilde{W})(\tilde{r} - r_f) \right] + AE \left[ U''(\tilde{W})(\tilde{r} - r_f)^2 \right]}{-AE \left[ U''(\tilde{W})(\tilde{r} - r_f)^2 \right]} \quad (51)$$

- Collecting terms in  $U''(\tilde{W})(\tilde{r} - r_f)$ , this can be rewritten as

## Implications for risk-taking with RRA

$$\eta = 1 + \frac{E \left[ U''(\tilde{W})(\tilde{r} - r_f) \{W_0(1 + r_f) + A(\tilde{r} - r_f)\} \right]}{-AE \left[ U''(\tilde{W})(\tilde{r} - r_f)^2 \right]} \quad (52)$$

$$= 1 + \frac{E \left[ U''(\tilde{W})(\tilde{r} - r_f) \tilde{W} \right]}{-AE \left[ U''(\tilde{W})(\tilde{r} - r_f)^2 \right]} \quad (53)$$

- The denominator in (53) is positive for  $A > 0$  by concavity. Therefore, if  $E \left[ U''(\tilde{W})(\tilde{r} - r_f) \tilde{W} \right] > 0$  then  $\eta > 1$  and the individual invests proportionally more in the risky asset with an increase in wealth.
- Can we relate this to the individual's risk aversion?

## Implications for risk-taking with DRRA

- Consider an individual whose *relative* risk aversion is decreasing in wealth.
- Then for  $A > 0$ , we again have  $W^h > W_0(1 + r_f)$ . When  $R_r(W) \equiv WR(W)$  is decreasing in wealth, this implies

$$W^h R(W^h) < W_0(1 + r_f) R(W_0(1 + r_f)) \quad (54)$$

- Multiplying both sides of (54) by  $-U'(W^h)(r^h - r_f)$ , which is a negative quantity, the inequality sign reverses:

$$W^h U''(W^h)(r^h - r_f) > -U'(W^h)(r^h - r_f) W_0(1 + r_f) R(W_0(1 + r_f)) \quad (55)$$

## Implications for risk-taking with DRRA

- For  $A > 0$ , we have  $W^l < W_0(1 + r_f)$ . If relative risk aversion is decreasing in wealth, this implies

$$W^l R(W^l) > W_0(1 + r_f) R(W_0(1 + r_f)) \quad (56)$$

- Multiplying (56) by  $-U'(W^l)(r^l - r_f)$ , which is positive, it equals

$$W^l U''(W^l)(r^l - r_f) > -U'(W^l)(r^l - r_f) W_0(1 + r_f) R(W_0(1 + r_f)) \quad (57)$$

- Inequalities (55) and (57) are the same whether the realization is  $\tilde{r} = r^h$  or  $\tilde{r} = r^l$ .
- Next, take expectations over all realizations of  $\tilde{r}$  to obtain

## Implications for risk-taking with DRRA

$$E \left[ \tilde{W} U''(\tilde{W})(\tilde{r} - r_f) \right] > -E \left[ U'(\tilde{W})(\tilde{r} - r_f) \right] W_0(1+r_f)R(W_0(1+r_f)) \quad (58)$$

- Since the first term on the right-hand side of inequality (58) is the FOC, the inequality simplifies to

$$E \left[ \tilde{W} U''(\tilde{W})(\tilde{r} - r_f) \right] > 0 \quad (59)$$

- Hence, decreasing relative risk aversion implies  $\eta > 1$  so an individual invests proportionally more in the risky asset as wealth increases.
- The opposite is true for increasing relative risk aversion:  $\eta < 1$  so that this individual invests proportionally less in the risky asset as wealth increases.

# Risk-taking with ARA/RRA

- The main results of this section can be summarized as:

## Risk Aversion

Decreasing Absolute

$$\frac{\partial A}{\partial W_0} > 0$$

Constant Absolute

$$\frac{\partial A}{\partial W_0} = 0$$

Increasing Absolute

$$\frac{\partial A}{\partial W_0} < 0$$

Decreasing Relative

$$\frac{\partial A}{\partial W_0} > \frac{A}{W_0}$$

Constant Relative

$$\frac{\partial A}{\partial W_0} = \frac{A}{W_0}$$

Increasing Relative

$$\frac{\partial A}{\partial W_0} < \frac{A}{W_0}$$

## Investment Behavior

# Conclusions

- We have shown:
  - Why expected utility, rather than expected value, is a better criterion for choosing and valuing assets.
  - What conditions preferences can satisfy to be represented by an expected utility function.
  - The relationship between a utility function,  $U(W)$ , and risk aversion.
  - How ARA/RRA affects the choice between risky and risk-free assets.