Chapter 5: Eigenvalues and Eigenvectors

1 Eigenvalues and Eigenvectors

Definition 1.1. If $\mathbf{A}$ is an $n \times n$ matrix, then a *nonzero* vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an **eigenvector** of A (or of the matrix operator T_A) if $\mathbf{Ax}$ is a scalar multiple of $\mathbf{x}$; that is,

$$\mathbf{Ax} = \lambda \mathbf{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of $\mathbf{A}$ (or of T_A), and $\mathbf{x}$ is said to be an eigenvector corresponding to λ.

Example 1.1. Let $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. Show that the vector $\mathbf{x} = [1, 2]^T$ is an eigenvector of $\mathbf{A}$.

In general, we will first compute the eigenvalue and then use it to compute each eigenvector corresponding to that eigenvalue.
To compute eigenvalue, we will use the following theorem.

Theorem 1.1. If $\mathbf{A}$ is an $n \times n$ matrix, then λ is an eigenvalue of $\mathbf{A}$ if and only if it satisfies the equation

$$det(\lambda \mathbf{I} - \mathbf{A}) = 0$$

This is called the **characteristic equation** of $\mathbf{A}$.

Remark:

Example 1.2. Find all eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

Example 1.3. Find all eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$.

Remarks:

- The **characteristic equation** of $\mathbf{A}$ is sometimes called **characteristic polynomial**. In particular, when $\mathbf{A}$ is an $n \times n$ matrix the determinant $p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$ is a polynomial of degree n:
$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + +c_{n-1}\lambda + c_n$$
where $c_1, ..., c_n$ are constants.

- Since a polynomial of degree n has at most n distinct roots, it follows that the characteristic equation of an $n \times n$ matrix $\mathbf{A}$ has at most n distinct solutions and consequently the matrix has fat most n distinct eigenvalues. Since some of these solutions may be complex numbers, it is possible for a matrix to have complex eigenvalues, even if that matrix itself has real entries.

Example 1.4. Find all eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$.

Theorem 1.2. If $\mathbf{A}$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of $\mathbf{A}$ are the **entries on the main diagonal** of $\mathbf{A}$.

Example 1.5. Find all eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} \pi & 0 & 0 \\ w & -2 & 0 \\ 1 & k & -1 \end{bmatrix}$, where k and w are constant numbers.

Theorem 1.3. Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are equivalent.

(a) λ is an eigenvalue of $\mathbf{A}$.

(b) λ is a solution of the characteristic equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

(c) The system of equations $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = 0$ has nontrivial solutions.

(d) There is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} =$
$lambda\mathbf{x}$.

A procedure for finding eigenvector(s) corresponding to a given eigenvalue λ
By definition, the eigenvectors of A corresponding to an eigenvalue λ are the nonzero vectors that satisfy the linear system

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = 0$$

Thus, we can find the eigenvectors of $\mathbf{A}$ corresponding to λ by finding the nonzero vectors in the solution space of this linear system. This solution space is called the **eigenspace** of $\mathbf{A}$ corresponding to λ.

Note that the **eigenspace** of $\mathbf{A}$ corresponding to λ is the same as

- the null space of the matrix $\lambda\mathbf{I} - \mathbf{A}$
- the kernel of the matrix operator $T_{\lambda\mathbf{I}-\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$
- the set of vectors for which $\mathbf{Ax} = \lambda\mathbf{x}$.

Example 1.6. Find bases for the eigenspaces of the matrix $\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

Example 1.7. Find bases for the eigenspaces of $\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Example 1.8. (Exercise) Prove the following theorem.

Theorem 1.4. A square matrix $\mathbf{A}$ is invertible if and only if $\lambda = 0$ is not an eigenvalue of $\mathbf{A}$.

Note: $\mathbf{A}$ is invertible if and only of $\det(\mathbf{A}) \neq 0$.

1.1 Eigenvalue of a Linear Operator

Definition 1.2. If $T : \mathbf{V} \to \mathbf{V}$ is a linear operator on a vector space $\mathbf{V}$, then a nonzero vector $\mathbf{x}$ in $\mathbf{V}$ is called an **eigenvector** of T if $T(\mathbf{x})$ is a scalar multiple of $\mathbf{x}$; that is,

$$T(\mathbf{x}) = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of T , and $\mathbf{x}$ is said to be an **eigenvector** corresponding to λ.

Example 1.9. Let $D : C^\infty \to C^\infty$ be the *differentiation operator* on the vector space of functions with continuous derivatives of all orders on the interval $(-\infty, \infty)$.
Show that $f(x) = e^{kx}$ is an eigenvector of D for any given constant k and find the corresponding eigenvalue to $f(x)$.

2 Diagonalization

Definition 2.1. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices, then we say that $\mathbf{B}$ is **similar** to $\mathbf{A}$ if there is an invertible matrix $\mathbf{P}$ such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Remarks:

- If $\mathbf{B}$ is similar to $\mathbf{A}$, then it is also true that $\mathbf{A}$ is similar to $\mathbf{B}$.
- In general, we will say that $\mathbf{A}$ and $\mathbf{B}$ are similar matrices if either is similar to the other.
- Any property that is preserved by a similarity transformation is called a similarity invariant and is said to be invariant under similarity. Table 1 lists the most important similarity invariants.

Property	**Description**
Determinant	A and $P^{-1}AP$ have the same determinant.
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible.
Rank	A and $P^{-1}AP$ have the same rank.
Nullity	A and $P^{-1}AP$ have the same nullity.
Trace	A and $P^{-1}AP$ have the same trace.
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial.
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues.
Eigenspace dimension	If λ is an eigenvalue of A (and hence of $P^{-1}AP$) then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

Definition 2.2. A square matrix $\mathbf{A}$ is said to be **diagonalizable** if it is similar to some diagonal matrix; that is,
if there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is diagonal.

In this case the matrix $\mathbf{P}$ is said to **diagonalize** $\mathbf{A}$.

The following theorem gives an approach for determining whether a matrix is diagonalizable and, if so, for finding a matrix $\mathbf{P}$ that will perform the diagonalization.

Theorem 2.1. If $\mathbf{A}$ is an $n \times n$ matrix, the following statements are equivalent.

(a) $\mathbf{A}$ is diagonalizable.

(b) $\mathbf{A}$ has n linearly independent eigenvectors.

Proof

Theorem 2.2. The following statements are true.

(a) If $\lambda_1, \lambda_, ..., \lambda_k$ are distinct eigenvalues of a matrix $\mathbf{A}$, and if $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ are corresponding eigenvectors, then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a linearly independent set.

(b) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

A Procedure for Diagonalizing an $n \times n$ Matrix

Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors.
One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n, then it is not.

Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix

$$\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n]$$

whose column vectors are the n basis vectors you obtained in Step 1.

Step 3. $\mathbf{P}^{-1}\mathbf{AP}$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_, ..., \lambda_n$ that correspond to the successive columns of $\mathbf{P}$.

Example 2.1. Find a matrix P that diagonalizes $\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Example 2.2. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalizable.

Exercise: Show that a triangular matrix with distinct entries on the main diagonal is diagonalizable.

2.1 Eigenvalues of Powers of a Matrix

Suppose that the eigenvalues and eigenvectors of $\mathbf{A}$ are known to be λ and $\mathbf{x}$, respectively, i.e.

$$\mathbf{A}\lambda = \lambda\mathbf{x}$$

Notice that we can compute the eigenvalues of $\mathbf{A}^2$ as follows.

$$\mathbf{A}^2\mathbf{x} =$$

The general case is given in the following theorem.

Theorem 2.3. Let k be a positive integer, λ be an eigenvalue of a matrix $\mathbf{A}$, and $\mathbf{x}$ is a corresponding eigenvector. Then λ^k is an eigenvalue of $\mathbf{A}^k$ and $\mathbf{x}$ is a corresponding eigenvectors.

Example 2.3. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$. Find the eigenvalues and eigenvectors of $\mathbf{A}^7$.

2.2 Computing Powers of a Matrix

One of the most efficient way to compute $\mathbf{A}^k$, particularly for large values of k, is to first diagonalize $\mathbf{A}$, which involves finding its eigenvalues and eigenvectors. We will see how these quantities are related to those of $\mathbf{A}^k$.
Suppose that $\mathbf{A}$ is a diagonalizable $n \times n$ matrix, that $\mathbf{P}$ diagonalizes $\mathbf{A}$:

Example 2.4. Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Find $\mathbf{A}^{13}$.

Definition 2.3. Let λ_0 be an eigenvalue of an $n \times n$ matrix $\mathbf{A}$.

- The dimension of the **eigenspace** corresponding to λ_0 is called the **geometric multiplicity** of λ_0.
- The number of times that $(\lambda - \lambda_0)$ appears as a factor in the characteristic polynomial of $\mathbf{A}$ is called the **algebraic multiplicity** of λ_0.

Theorem 2.4. (Geometric and Algebraic Multiplicity)
Let $\mathbf{A}$ be a square matrix.

(a) For every eigenvalue of $\mathbf{A}$, the **geometric multiplicity** is less than or equal to the **algebraic multiplicity**.

(b) A is diagonalizable if and only if the **geometric multiplicity** of every eigenvalue is equal to the **algebraic multiplicity**.

Example 2.5. For each of the following matrix, find geometric multiplicity and algebraic multiplicity, and check if it is diagonalizable by using the previous theorem.

(a) $\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

(b) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$