Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks)

Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} \; E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$J(W_t, I_t, t) = \max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1} \delta^s \left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + \delta^T \left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

at T, $J(W_T, T) = E_T[B(W_T, T)] = B(W_T, T)$

Using backward dynamic program

$$J(W_{T-1}, T-1) = \max_{C_{T-1},\{w_i, T-1\}} E_{T-1}[U(C_{T-1}, T-1) + B(W_T, T)]$$

$$= \max_{C_{T-1},\{w_i, T-1\}} U(C_{T-1}, T-1) + E_{T-1}[B(W_T, T)]$$

$$= \max_{C_{T-1},\{w_i, T-1\}} U(C_{T-1}, T-1) + E_{T-1}[B(S_{T-1}R_{T-1})]$$

Recall: $S_{T-1} = W_{T-1} + y_{T-1} - C_{T-1}$

$R_{T-1} = R_{f,T-1} + \Sigma_{i=1}^{n} w_{i,T-1}(R_{i,T-1} - R_{f,T-1})$

FOC:

$$\frac{\partial J}{\partial C_{T-1}} = \delta^{T-1} \cancel{(1-\gamma)} \frac{C_{T-1}^{-\gamma}}{\cancel{(1-\gamma)}} - E_{T-1}\left[\delta^T \cancel{(1-\gamma)} \frac{W_T^{-\gamma}}{\cancel{(1-\gamma)}} \cdot R_{T-1}\right] = 0 \quad - 1$$

$$\frac{\partial J}{\partial w_{i,T-1}} = E_{T-1}\left[\delta^T \cancel{(1-\gamma)} \frac{W_T^{-\gamma}}{\cancel{(1-\gamma)}} (R_{i,T-1} - R_{f,T-1})\right] = 0 \quad - 2$$

Using 2, 1 can be rewritten

$$\delta^{T-1} C_{T-1}^{*-\gamma} = E_{T-1}[\delta^T W_T^{-\gamma} (R_{T-1})]$$

$$\delta^{T-1} C_{T-1}^{*-\gamma} = R_{f,T-1} E_{T-1}[\delta^T W_T^{-\gamma}] = E_{T-1}[\delta^T (S_{T-1}R_{T-1})^{-\gamma} (R_{T-1})] \quad - 3$$

$$= E_{T-1}\left[\delta^T \frac{R_{T-1}}{(S_{T-1}R_{T-1})^\gamma}\right]$$

$$= E_{T-1}\left[\delta^T \frac{(R_{T-1})^{1-\gamma}}{(W_{T-1} - C_{T-1})^\gamma}\right]$$

$$C_{T-1}^{*\gamma} = \frac{1}{1+\delta} W_{T-1}^{\gamma} (R_{T-1})^{1-\gamma}$$

∴ consumption for this investor is a fixed proportion of wealth and independent of investment opportunities.

Score.....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1},T-1) = \max_{c_{T-1},\{w_{i},T-1\}} E_{T-1}\left[U(c_{T-1},T-1) + B(W_T,T)\right]$$

$$= \max_{c_{T-1},\{w_{i},T-1\}} U(c_{T-1},T-1) + E_{T-1}\left[B(W_T,T)\right]$$

$$= \max_{c_{T-1},\{w_{i},T-1\}} U(c_{T-1},T-1) + E_{T-1}\left[B(S_{T-1}R_{T-1})\right]$$

Substitute c^*_{T-1} and w^*_{T-1} back into $J(W_{T-1},T-1)$ and differentiate with respect to W_{T-1}

$$J_W = \left[\delta^{T-1} c_{T-1}^{*\gamma}\right]\left(\frac{\partial c^*_{T-1}}{\partial W_{T-1}}\right) + E_{T-1}\left[\delta^T W_T^{-\gamma}\left(\frac{dW_T}{dW_{T-1}}\right)\right]$$

$$= \left[\delta^{T-1} c_{T-1}^{*\gamma}\right]\left(\frac{\partial c^*_{T-1}}{\partial W_{T-1}}\right) + E_{T-1}\left[\delta^T W_T^{-\gamma}\left(\frac{\partial W_T}{\partial W_{T-1}} + \sum_{i=1}^{n}\frac{\partial W_T}{\partial w^*_{i,T-1}}\frac{\partial w^*_{i,T-1}}{\partial W_{T-1}} + \frac{\partial W_T}{\partial c^*_{T-1}}\frac{\partial c^*_{T-1}}{\partial W_{T-1}}\right)\right]$$

$$= \left[\delta^{T-1} c_{T-1}^{*\gamma}\right]\left(\frac{\partial c^*_{T-1}}{\partial W_{T-1}}\right) + E_{T-1}\left[\delta^T W_T^{-\gamma}\left(\sum_{i=1}^{n}\left[R_{i,T-1} - R_{f,T-1}\right]S_{T-1}\frac{\partial w^*_{i,T-1}}{\partial W_{T-1}} + R_{T-1}\left(1 - \frac{\partial c^*_{T-1}}{\partial W_{T-1}}\right)\right)\right] \quad - (4)$$

From (2) and (3), (4) simplifies to

$$J_W = \cancel{\left[\delta^{T-1} c_{T-1}^{*\gamma}\right]\left(\frac{\partial c^*_{T-1}}{\partial W_{T-1}}\right)} - \cancel{E_{T-1}\left[\delta^T W_T^{-\gamma} \cdot R_{T-1}\right]\frac{\partial c^*_{T-1}}{\partial W_{T-1}}} + E_{T-1}\left[\delta^T W_T^{-\gamma} \cdot R_{T-1}\right]$$

$$= E_{T-1}\left[\delta^T W_T^{-\gamma} \cdot R_{T-1}\right]$$

Then (3) can be rewritten as

$$J_W(W_{T-1},T-1) = U_c(c^*_{T-1},T-1) = \delta^{T-1} c_{T-1}^{*\gamma}$$

Score....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

at T-2,

$$J(W_{T-2}, T-2) = \max_{C_{T-2}, \{w_i, T-2\}} U(C_{T-2}, T-2) + E_{T-2}\left[U(C_{T-1}, T-1) + B(W_T, T)\right]$$

$$= \max_{C_{T-2}, \{w_i, T-2\}} \delta^{T-2} U(C_{T-2}) + \delta^{T-1} E_{T-2}\left[(1+\delta) W_{T-1}\right]$$

$$U_C(C^*_{T-2}, T-2) = E_{T-2}\left[J_W(W_{T-1}, T-1) R_{T-2}\right]$$

$$\frac{\delta^{T-2}}{C^{\gamma}_{T-2}} = (1+\delta)\delta^{T-1} E_{T-2}\left[\frac{R_{T-1}}{(S_{T-1}R_{T-1})^{\gamma}}\right]$$

$$= \frac{(1+\delta)\delta^{T-1}}{(W_{T-1} - C_{T-1})^{\gamma}} (R_{T-1})^{1-\gamma}$$

$$C^*_{T-2} = \frac{1}{1+\delta+\delta^2} W^{\gamma}_{T-1} (R_{T-1})^{1-\gamma}$$

$\therefore$ consumption for this investor is a fixed proportion of wealth and independent of investment opportunities.

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

at T-2, $J(W_{T-2}, T-2) = \max_{C_{T-2}, \{w_i, T-2\}} U(C_{T-2}, T-2) + E_{T-2}\left[U(C_{T-1}, T-1) + B(W_T, T)\right]$

$= \max_{C_{T-2}, \{w_i, T-2\}} \delta^{T-2} U(C_{T-2}) + \delta^{T-1} E_{T-2}\left[(1+\delta) W_{T-1}\right]$

$U_c(C^*_{T-2}, T-2) = E_{T-2}\left[J_W(W_{T-1}, T-1) R_{T-2}\right]$

$= R_{f, T-2} E_{T-2}\left[J_W(W_{T-1}, T-1)\right]$

$= J_W(W_{T-2}, T-2)$