Assignment 4

DUE DATE: Tuesday 9nd, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

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Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean (μ) = 0 and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$ - characteristic equation

$x^2-1.5x+0.9=0$ - reverse characteristic equation

modulus < 1 or $R=\sqrt{a^2+b^2}$ < $\therefore$ model is stationary

eigenvalues : $\lambda_i = \dfrac{\phi_1 \pm \sqrt{\phi_1^2+4\phi_2}}{2}$

$\lambda_i = \dfrac{-(-1.5) \pm \sqrt{(-1.5)^2-4(1)(0.9)}}{2}$

$\lambda_1 = 0.75+0.5809i$

$\lambda_2 = 0.75-0.5809i$

$R=\sqrt{0.75^2+0.5809^2} = 0.9487$ which is less than 1.

The model is a weak stationary #

Question 1.2 (10 points)

Your score......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

$$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$$
$$R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25+\varepsilon_t$$
$$R_t = 1.5R_{t-1} - 0.9R_{t-2} + 0.25+\varepsilon_t$$

=0, Guassian white noise ~ $(0,\sigma^2)$

$$E[R_t] = 1.5E[R_{t-1}] - 0.9E[R_{t-2}] + 0.25 + E[\cancel{\varepsilon_t}]$$

$\therefore$ Because the model is weak stationary, $E[r_t] = E[r_{t-1}] = E[r_{t-2}] = \mu$

$$E[R_t] = 1.5E[R_t] - 0.9E[R_t] + 0.25$$
$$E[R_t] + 0.9E[R_t] - 1.5E[R_t] = 0.25$$
$$0.4E[R_t] = 0.25$$
$$E[R_t] = 0.625 \#$$ - unconditional mean #

$$E[R_t|F_{t-1}] = 1.5E[R_{t-1}] - 0.9E[R_{t-2}] + 0.25 + \underbrace{\cancel{E[\varepsilon_t|F_{t-1}]}}_{=0}$$

$$E[R_t|F_{t-1}] = 0.25 + 1.5E[R_{t-1}] - 0.9E[R_{t-2}]$$ - conditional mean #

Question 1.3 (10 points) Your score......................

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$

$R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25+\varepsilon_t$

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

$(R_t-\mu) = 0.25 + 1.5(R_{t-1}-\mu) - 0.9(R_{t-2}-\mu) + \varepsilon_t$

$(R_t-\mu)^2 = 0.25^2 + 1.5^2(R_{t-1}-\mu)^2 - 0.9^2(R_{t-2}-\mu)^2 + \varepsilon_t^2 + 2(0.25)(1.5)(R_{t-1}-\mu)$
$+2(0.25)(-0.9)(R_{t-2}-\mu)$
$+2(0.25)(\varepsilon_t)$
$+2(1.5)(-0.9)(R_{t-1}-\mu)(R_{t-2}-\mu)$
$+2(1.5)(\varepsilon_t)(R_{t-1}-\mu)$
$+2(-0.9)(R_{t-2}-\mu)(\varepsilon_t)$

$var(R_t) = E[(R_t-\mu)^2]$, that take $E(\cdot)$
and the model is weak stationary
$var(R_t) = var(R_{t-1}) = var(R_{t-2})$

$$\left.\begin{aligned} var(R_t) = 0.0625 + 2.25var(R_t) + 0.81var(R_t) + var(\varepsilon_t) &+ 0.75E[\underbrace{\cancel{R_{t-1}}-\mu}_{=\mu-\mu=0}] \\ &- 0.45E[R_{t-2}-\mu] \\ &+ 0.50E[\underbrace{\cancel{\varepsilon_t}}_{=0}] \end{aligned}\right\} = 0 \text{ from properties of stationary}$$

$$- 2.7E[(R_{t-1}-\mu)(R_{t-2}-\mu)]$$

$$\left.\begin{aligned} &+3.0E[\varepsilon_t(R_{t-1}-\mu)] \\ &-1.8E[(\varepsilon_t)(R_{t-2}-\mu)] \end{aligned}\right\} = 0$$

$$var(R_t) = \frac{\sigma^2 + 2(\phi_1)(\phi_2)\gamma_1}{1-\phi_1^2-\phi_2^2} = \frac{0.0625+0.25-2.7\gamma_1}{1-(1.5)^2-(0.9)^2} = \frac{0.3125-2.7\gamma_1}{-2.06} \text{ - unconditional variance}$$

$cov(r_t, r_{t-j})$

$$E[(r_t-\mu)(r_{t-j}-\mu)] = 0.25E[\underbrace{\cancel{r_{t-j}}-\mu}_{=0}] + 1.5E[\underbrace{(r_{t-1}-\cancel{\mu})(r_{t-j}-\mu)}_{=\gamma_{j-1}}]$$
$$-0.9E[\underbrace{(r_{t-2}-\cancel{\mu})(r_{t-j}-\mu)}_{\gamma_{j-2}}] + E[(\varepsilon_t)(\underbrace{\cancel{r_{t-j}}-\mu}_{=0})]$$

$\gamma_j = 1.5\gamma_{j-1} - 0.9\gamma_{j-2}$, $\gamma_1 = 1.5\gamma_0 - 0.9\gamma_1$

$\gamma_1 = 0.8\, var(r_t)$

$$var(R_t) = \frac{0.3125-2.7(0.8var(R_t))}{-2.06}$$

$-2.06var(R_t) = 0.3125 - 2.16var(R_t)$

$0.1var(R_t) = 0.3125$

$var(R_t) = 3.125$ - unconditional variance #

conditional variance

$$var(r_t|F_{t-1}) = \underbrace{1.5^2(var\, r_{t-1}|F_{t-1}) + 0.9^2 var(r_{t-2}|F_{t-1})}_{\text{no variation of past return}} + var(\varepsilon_t) + \overbrace{2cov(\phi_0,\phi_1,\phi_2|F_{t-1})}^{=0}$$

$var(r_t|F_{t-1}) = \sigma^2 = 0.25$ #

Question 1.4 (10 points) | Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

AR(2) : $\rho_j = 1.5\rho_{j-1} - 0.9\rho_{j-2}$ # $\therefore \gamma_j = 1.5\gamma_{j-1} - 0.9\gamma_{j-2}$

AR(2) : $\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}$

ρ_l when $l=1$

$\rho_1 = 1.5\rho_0 - 0.9\rho_1$

$1.9\rho_1 = 1.5\rho_0$

$\rho_1 = 0.8$ #

ρ_l when $l=2$

$\rho_2 = 1.5\rho_1 - 0.9\rho_0$

$\rho_2 = 1.5(0.8) - 0.9$

$\rho_2 = 0.3$ #

Question 1.5 (10 points) | Your score......................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$

$R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25+\varepsilon_t$

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

1-Step Ahead

(1) point estimator $\hat{R}_{1000}^{(1)} = 0.25 + 1.5R_{999} - 0.9R_{998}$

$\hat{R}_{1000}^{(1)} = 0.25 + 1.5(0.02) - 0.9(0.03)$

$\hat{R}_{1000}^{(1)} = 0.253$

(2) forecasting error : actual - estimate : $R_{1001} - \hat{R}_{1000}^{(1)} = \varepsilon_{1001}$

(3) Variance of forecasting : $var(\varepsilon_t^{(1)}|\cdot) = var(a_{h+1}|\cdot) = \sigma_a^2 = 0.25$

(4) Interval forecasting using 95% CI : $0.253 \pm 1.96\sqrt{0.25}$

$= (-0.727, 1.233)$ #

2-step Ahead

(1) Point estimator : $\hat{R}_{1000}^{(2)} = 0.25 + 1.5\hat{R}_{1000}^{(1)} - 0.9R_{1000}$

$= 0.25 + 1.5(0.253) - 0.9(0.01)$

$= 0.6205$

(2) fore casting error : $R_{1002} - \hat{R}_{1000}^{(2)} = 1.5\varepsilon_{1001} + \varepsilon_{1002}$

(3) variance of error : $(1.5^2)\sigma_a^2 + \sigma_a^2 + 2cov(\underbrace{\cancel{\varepsilon_{1001}}, \varepsilon_{1002}}_{=0})$

$var(\varepsilon_t^{(2)}|\cdot) = (2.25)(0.25) + 0.25 = 0.8125$

(4) Interval forecasting at 95% CI $= 0.6205 \pm 1.96\sqrt{0.8125}$

$(-1.146, 2.387)$ #