Definition 6.2.8: (Algebra of Functions)

Let f and g be two functions. The sum, $f+g$, the difference, $f-g$, the product, fg, and the quotient, $\frac{f}{g}$ are functions defined as follows:

1. Sum: $(f+g)(x)=f(x)+g(x)$

2. Difference: $(f-g)(x)=f(x)-g(x)$

3. Product: $(fg)(x)=f(x)\cdot g(x)$

4. Quotient: $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}, \quad g(x)\neq 0$

The domains of the sum, difference, and product functions are the set of all real numbers common to the domains of both f and g. The domain of $\frac{f}{g}$ includes all real numbers common to the domains of f and g for which $g(x)\neq 0$.

Example 6.2.9: Recall Example 6.2.2: Let $A=\{0,1,2\}$ and $B=\{0,1,4\}$. Let f and g be functions defined by

$f=\{(x,y)\in A\times B \mid y=x^2\}$

$g=\{(x,y)\in A\times B \mid 2x+y=2 \vee 2x+y=4\}$

Find $f+g, f-g, g-f, fg, \frac{f}{g}$, and $\frac{g}{f}$

Example 6.2.10: Let f and g be functions defined by $f(x)=\sqrt{x+3}$ and $g(x)=\dfrac{1}{x^2-4}$. Find $f+g,\ f-g,\ fg,\ \dfrac{f}{g}$, and $\dfrac{g}{f}$

Example 6.2.11: Let $f(x) = 2 - x$ and $g(x) = \begin{cases} x^2 + 1, & 0 \le x < 1 \\ x - 3, & x \ge 1 \end{cases}$. Find $f - g$ and $\dfrac{f}{g}$.

Definition 6.2.9: (Composition of Functions)

If f and g are functions which $R_f \cap D_g \neq \varnothing$, then the composite function of f and g, denoted by $g \circ f$, is defined by $(g \circ f)(x) = g(f(x))$ where $x \in D_f$.

Example 6.2.12: Let $A = \{1,2,3,4,5\}$, $B = \{-3,-2,-1,0,6\}$, $C = \{-4,7,8,9\}$, and $f : A \to B$, $g : B \to C$ where $f = \{(1,-3),(2,-2),(3,0),(4,-1),(5,-2)\}$ and $g = \{(-3,7),(-2,9),(-1,7),(0,7),(6,8)\}$. Find $g \circ f$ if possible.

Example 6.2.13: Let $f(x)=\dfrac{2}{\sqrt{x-1}}$ and $g(x)=x^2+3$. Find $g\circ f$ and $f\circ g$ if possible.

Example 6.2.14: Recall Example 6.2.7. Let $f(x)=1+\sqrt{x-2}$ then $f^{-1}(x)=2+(x-1)^2$. Find $f\circ f$ and $f\circ f^{-1}$ if possible.

Example 6.2.15: Let $f(x) = x^2 + 5$ and $g(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{1}{x}, & 2 \le x < 3 \end{cases}$.

Find $f \circ g$ and $g \circ f$ if possible.

Example 6.2.16: Let f be a function defined by $f(x) = -x^2 - 1$ and let g be a function defined by $g(x) = \begin{cases} 1 - x & , x < 0 \\ x^2 + 1 & , x \ge 0 \end{cases}$. Find $g \circ f$ if possible.

Example 6.2.17: Let $f(x)=\begin{cases} x-1 & ,0\le x<2 \\ x^2+1 & ,x\ge 2 \end{cases}$ and $g(x)=\begin{cases} x+3 & ,-\frac{1}{2}\le x<0 \\ \frac{1}{x+1} & ,x\ge 0 \end{cases}$.

Find $g\circ f$ if possible.

6.3 Types of Functions

In this section, we will be introduced several types of functions. In general, functions have two types: algebraic functions and transcendental functions.

6.3.1 **Algebraic functions**

There are two types of algebraic functions.

1) **Polynomial functions**: Polynomial function is a function of the form $P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1x+a_0$ where n is a positive integer or zero and $a_0,a_1,a_2,...,a_n$ are real numbers. Since the polynomial function is a function from a set of real numbers to real numbers, therefore the domain of polynomial functions is a set of real numbers.

2) **Rational functions**: Rational function is a function of the form $y=\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x)\neq 0$. For example, $\dfrac{x^2+1}{x^2-x+2},\dfrac{x-1}{2x^2+x+1},\dfrac{1}{x-3}$,etc. The domain of a rational function is a set of any real numbers where $Q(x)\neq 0$, that is $D_f=\{x\in\mathbb{R}\,|\,Q(x)\neq 0\}$.

6.3.2 **Transcendental functions**

Transcendental function is a function that is not an algebraic function.

1) **Exponential functions base *a***: An exponential function is the function of the form $y=ba^{P(x)}$ where a and b are real numbers, $a>0$, $a\neq 1$, $b\neq 0$, and $P(x)$ is a polynomial function. For example, $y=2^{x+3}$, $y=-3^{x^2+1}$, $y=e^{2x}$, etc. The domain of an exponential function base a is a set of real numbers.

2) **Logarithmic functions:** A logarithmic function is an inverse function of an exponential function, that is the logarithmic function that is defined by $y=\log_a x$ will be an inverse function of the exponential function base a which is $y=a^x$ where $a>0$ and $a\neq 1$. For example, $y=\log_2 x$, $y=\log x$, and $y=\ln x$, etc.

Note: (i) Logarithmic function base 10 is denoted by $y=\log x$.

(ii) The natural logarithm is the logarithmic function base e and is denoted by $y=\ln x$.

6.4 Applications of Functions

Example 6.4.1: A rectangular box has a square base and the length of each side of the base is twice the height of the box. Let $f(x)$ be a function of its volume and let x be the length of each side of the base (in centimeter).

1) Find $f(x)$.
2) If the height of this box is 6 cm, then find the volume of the box.

Example 6.4.2: A company produces two products called A and B. They produce product A and product B in a ratio of 1:3. The cost of producing both products is the same, which is $1-3x^2-x$ (in Baht) where *x* is number of product A that is produced by this company. The sale manager set the price for product A and B equal to 25 Baht and 23 Baht, respectively. Assume that the company sold all the products.

1) Find the function of the profit from selling both products in term of *x*.
2) If the company produces 1,050 units of product B, find the profit from selling both products.

6.5 Partial Fractions Decomposition

Consider a rational function

$$f(x)=\frac{P(x)}{Q(x)}$$

where P and Q are both polynomials. A rational expression is proper if the degree of the numerator is lower than the degree of the denominator; otherwise it is improper. An improper rational expression is the sum of a polynomial and a proper rational expression. The division statement is

$$f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

For example

$$\frac{x^3+x}{x+2}=x^2-2x+5-\frac{10}{x+2}$$

where x^2-2x+5 is the quotient and -10 is the remainder upon division of the denominator into the numerator.

Partial fraction decomposition is the process of rewriting a rational expression as the sum of a quotient polynomial plus partial fractions. If the rational expression is proper, the quotient will be zero.

Example 6.5.1: Find the partial fraction decomposition of $\frac{x^3+2x^2-1}{x-2}$.

After finding the quotient, the next step in decomposing a rational expression is factoring the denominator. It can be shown that any polynomial Q (the denominator) can be factored as a product of linear factors (of the form $ax+b$) and irreducible quadratic factors (of the form ax^2+bx+c, where $b^2-4ac<0$). For instance, if $Q(x)=x^4-16$, we could factor it as

$$Q(x)=x^4-16=\left(x^2+4\right)\left(x^2-4\right)=\left(x^2+4\right)(x+2)(x-2)$$

The third step is to express the proper rational function $R(x)/Q(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{\left(ax^2+bx+c\right)^j}$$

<u>CASE I</u>: The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, ..., A_k$ such that

$$\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k} \qquad (1)$$

Example 6.5.2: Find the partial fraction decomposition of

(a) $\dfrac{x-9}{x^2+3x-10}$

(b) $\dfrac{3x^2-7x-2}{x^3-x}$

<u>CASE II</u>: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$ in Equation (1), we would use

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r} \tag{2}$$

By way of illustration, we could write

$$\frac{x^3+4}{x^2(x+1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}$$

Example 6.5.3: Find the partial fraction decomposition of

(a) $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

(b) $\dfrac{x-2}{x^2(x-1)^2}$

<u>CASE III</u>: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then, in addition to the partial fractions in Equation (1) and (2), the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c} \tag{3}$$

where A and B are constants to be determined. For instance, the function given by $x/\left[(x-3)(x^2+1)(x^2+2)\right]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-3)(x^2+1)(x^2+2)}=\frac{A}{x-3}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+2}$$

Example 6.5.4: Find the partial fraction decomposition of

(a) $\dfrac{3x^2-4x+3}{x^3+x}$

(b) $\dfrac{6x^2-3x+1}{(4x+1)(x^2+1)}$

<u>CASE IV</u>: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction (3), the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r} \qquad (4)$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the term in (4) can be integrated by first completing the square.

Example 6.5.5: Find the partial fraction decomposition of $\dfrac{6x^2-15x+22}{(x+3)(x^2+2)^2}$.