Solow Growth Model : Steady-State Derivation.

Given $\quad Y = F(K, L) = AK^a L^b \quad$ where $a + b = 1$

and $0 < a < 1$

$0 < b < 1$

And $\quad \Delta K = sY - dK.$

Write everything in terms of per capita :

$$k = \frac{K}{L} \Rightarrow y = \frac{Y}{L} = \frac{AK^a L^b}{L} = \frac{AK^a L^{1-a}}{L} = \frac{AK^a}{L^a}$$

$$\therefore \quad y = A\left(\frac{K}{L}\right)^a = Ak^a \qquad \text{—— (1)}$$

From $\quad \Delta K = sY - dK$

$$\frac{\Delta K}{K} = \frac{sY}{K} - \frac{d\cancel{K}}{\cancel{K}} = \frac{sY}{K} - d \qquad \text{—— (2)}$$

Since $k = \frac{K}{L}$, $\quad \ln(k) = \ln\left(\frac{K}{L}\right) = \ln(K) - \ln(L)$

Take total differentiation :

$$d[\ln(k)] = d[\ln(K) - \ln(L)]$$

$$\frac{1}{k} \cdot dk = \frac{1}{K} dK - \frac{1}{L} \cdot dL \qquad \text{(Note ; } d = \text{differential). or derivative)}$$

Since $dk \simeq \Delta k$, we have

$$\frac{\Delta k}{k} = \frac{\Delta K}{K} - \frac{\Delta L}{L} \qquad \text{—— (3)}$$

From (2) & (3),

$$\frac{\Delta k}{k} = \underbrace{\left[\frac{sY}{K} - d\right]}_{\frac{\Delta K}{K}} - \frac{\Delta L}{L}$$

Sub in $Y = AK^a L^b$:

depreciation rate

$$\frac{\Delta k}{k} = \frac{s}{k} \cdot (AK^a L^b) - d - n \quad \left(\text{since } \frac{\Delta L}{L} = n\right)$$

$$\frac{\Delta k}{k} = s \cdot AK^{a-1} L^{1-a} - (d+n).$$

$$\frac{\Delta k}{k} = s \cdot A\left(\frac{K}{L}\right)^{a-1} - (d+n) = s \cdot A k^{a-1} - (d+n). \rightarrow *$$

At steady-state, $\Delta k = 0$,

$$\frac{\Delta k}{k} = 0 = sAk_{ss}^{a-1} - (d+n)$$

$$\Rightarrow sAk_{ss}^{a-1} = (n+d)$$

$$k_{ss}^{a-1} = \frac{(n+d)}{sA}$$

$$\therefore \quad k_{ss} = \left[\frac{n+d}{sA}\right]^{1/(a-1)}$$

or $$\boxed{k_{ss} = \left[\frac{sA}{(n+d)}\right]^{1/(1-a)}}$$ since $0 < a < 1$

At s.s, $$y_{ss} = Ak_{ss}^a = A\left[\frac{sA}{(n+d)}\right]^{a/(1-a)}.$$

Also, at s.s., $\boxed{g_y = 0}$ because $g_y = \frac{\Delta y}{y}$ and $\Delta y_{ss} = 0$ $(\because \Delta k_{ss} = 0)$

But. $$g_Y = \frac{\Delta Y}{Y} = \frac{\Delta y}{y} + \frac{\Delta L}{L}$$, since $y = \frac{Y}{L} \Rightarrow \frac{\Delta y}{y} = \frac{\Delta Y}{Y} - \frac{\Delta L}{L}$

$$\therefore \quad \boxed{g_Y = \frac{\Delta Y}{Y} = n}$$

→ Read explanation on slide #17 in lecture #3.