Solution: Quiz 5

1. Define $H : \mathbb{Z}^+ \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}^+$ as follows:

$$H(x, y) = (\ y + 1,\ 3x^2\ ) \qquad \text{for } (x, y) \in \mathbb{Z}^+ \times \mathbb{Z},$$

where $\mathbb{Z}^+$ is the set of all positive integers and $\mathbb{Z}$ is the set of all integers.

(a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

Solution:

(a) Is H one-to-one? Prove or give a counterexample.
Answer: Yes, H is one-to-one. Let $(x_1, y_1), (x_2, y_2)$ be some elements in the domain $\mathbb{Z}^+ \times \mathbb{Z}$. We want to show that if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$.
Suppose $H(x_1, y_1) = H(x_2, y_2)$. Then

$$(\ y_1 + 1,\ 3x_1^2\ ) = (\ y_2 + 1,\ 3x_2^2\ )$$

or equivalently, $y_1 + 1 = y_2 + 1$ and $3x_1^2 = 3x_2^2$. I.e.,

$$y_1 + 1 = y_2 + 1 \qquad \Rightarrow \qquad y_1 = y_2$$

and

$$3x_1^2 = 3x_2^2 \quad \Leftrightarrow \quad x_1^2 = x_2^2 \quad \Leftrightarrow \quad x_1 = \pm x_2 \quad \Leftrightarrow \quad x_1 = x_2,$$

(where we have used the fact that both x_1 and x_2 are in $\mathbb{Z}^+$ and so $x_1^2 = x_2^2 \Leftrightarrow x_1 = x_2$ for $x_1, x_2 > 0$). That is, $H(x_1, y_1) = H(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$ and therefore, H is one-to-one.

(b) Is H onto? Prove or give a counterexample.
Answer: No, H is not onto. Notice that if we pick (u, v) from the co-domain $\mathbb{Z}^+ \times \mathbb{Z}$ and suppose that there is (x, y) in the domain such that $H(x, y) = (u, v)$, then

$$(u, v) = H(x, y) = (\ y + 1,\ 3x^2\ )$$

or we must have

$$u = y + 1 \quad \Rightarrow \quad y = u - 1$$

and

$$v = 3x^2 \quad \Rightarrow \quad x = \pm\sqrt{\frac{v}{3}}.$$

So, the following is a counterexample for this. If we choose $(u, v) = (1, 1)$ from the co-domain $\mathbb{Z} \times \mathbb{Z}^+$, then, in order to have $H(x, y) = (1, 1)$, we must set $x = \pm\sqrt{\frac{1}{3}}$, and set $y = 1 - 1 = 0$. However, $x = \pm\sqrt{\frac{1}{3}}$ are not in $\mathbb{Z}^+$ and therefore, we **cannot** find any element (x, y) from the domain $\mathbb{Z}^+ \times \mathbb{Z}$ such that $H(x, y) = (1, 1)$. That is, H is not onto.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.
Answer: No, H is not bijective because it is not onto. So we cannot find its inverse function.