Exericse Solution: Solving Inequality (Part II)

1. Let a and b be real numbers with $0 < b < a < 1$. Determine whether each of the following inequalities is true or not. Explain your answer.

   (a) $\frac{(b+1)^3}{|a|} < \frac{(a+1)^3}{b}$

   (b) $\frac{b}{a} < \frac{a^3-1}{b^3-1}$

   Solution:

   (a) $\frac{(b+1)^3}{|a|} < \frac{(a+1)^3}{b}$ is **true**.
   From $0 < b < a < 1$, by adding 1 throughout the inequalities, we have

   $$0 + 1 < b + 1 < a + 1 < 1 + 1$$

   and so, $0 < b + 1 < a + 1$

   $$0 < (b + 1)^3 < (a + 1)^3. \quad (1)$$

   Since $a > 0$, $|a| = a$. Since $a, b > 0$, then $b < a$ implies $\frac{1}{ab}b < \frac{1}{ab}a$ or

   $$0 < \frac{1}{|a|} < \frac{1}{b} \quad (2)$$

   From (1) and (2),

   $$\frac{(b+1)^3}{|a|} < \frac{(a+1)^3}{b}.$$

   (b) $\frac{b}{a} < \frac{a^3-1}{b^3-1}$ is **false**.

   Since $a, b > 0$, $b < a$ implies

   $$\frac{b}{a} < 1.$$

   From $0 < b < a < 1$, we have $0 < b^3 < a^3 < 1$ and by subtracting 1 throughout the inequalities, we have

   $$-1 < b^3 - 1 < a^3 - 1 < 0.$$

   Since $b^3 - 1 < 0$, when we divide $b^3 - 1$ throughout the equality $b^3 - 1 < a^3 - 1$ we have

   $$\frac{a^3-1}{b^3-1} < 1.$$

   That is, we only have $\frac{b}{a} < 1$ and $\frac{a^3-1}{b^3-1} < 1$ and we cannot conclude that $\frac{b}{a} < \frac{a^3-1}{b^3-1}$.
   A counterexample is when $a = \frac{4}{5}$ and $b = \frac{3}{4}$. We noticed that $0 < \frac{3}{4} < \frac{4}{5} < 1$ and so $0 < b < a < 1$,

   $$\frac{b}{a} = \frac{3/4}{4/5} = \frac{15}{16} = 0.9375$$

   and

   $$\frac{a^3-1}{b^3-1} = \frac{(4/5)^3-1}{(3/4)^3-1} = 2063/2444 \approx 0.8441$$

   Since $0.9375 \not< 0.844$, we can find a and b with $0 < b < a < 1$ such that $\frac{b}{a} \not< \frac{a^3-1}{b^3-1}$. ■

2. Let x and y be real numbers with $y > x > 1$. Show that

$$y(y-2) > |x-1|^2 - 1.$$

Solution: From $y > x > 1$, subtracting 1 throughout these inequalities gives

$$y - 1 > x - 1 > 0,$$

and since both $y-1$ and $x-1$ are positive numbers, we can square both sides of $y-1 > x-1$:

$$(y-1)^2 > (x-1)^2$$

or equivalently,

$$y^2 - 2y + 1 > (x-1)^2 \quad \Leftrightarrow \quad y^2 - 2y > (x-1)^2 - 1 \quad \Leftrightarrow \quad y(y-2) > (x-1)^2 - 1$$

Since $(x-1)^2 = |x-1|^2 \ \forall x \in \mathbb{R}$, then we have

$$y(y-2) > |x-1|^2 - 1.$$

■

3. Find the solution set for each of following inequalities.

(a)
$$\frac{x^2-1}{x} < \frac{x+1}{2x}$$

(b)
$$2 \leq \left|\frac{x-1}{x}\right| \leq 7$$

(c)
$$\frac{x^2+|x|+1}{x^7+x^5+x^2+1} \leq 0$$

(d)
$$\frac{|x+3|-2}{5} + \frac{1}{|x-1|+1} \leq 1$$

(e)
$$\frac{|x-1|-x^2-1}{5-|x+3|} \geq 0$$

Solution:

(a)
$$\frac{x^2-1}{x} < \frac{x+1}{2x}$$

Solution:

$$\begin{aligned}
\frac{x^2-1}{x} &< \frac{x+1}{2x} \\
\frac{x^2-1}{x} - \frac{x+1}{2x} &< 0 \\
\frac{2x^2-2-x-1}{x} &< 0 \\
\frac{2x^2-x-3}{x} &< 0 \\
\frac{(x+1)(2x-3)}{x} &< 0
\end{aligned}$$

Notice that $(x+1)(2x-3) =$ when $x = -1, 3/2$ and $x = 0$ when $x = 0$. To obtain the subintervals, consider $x = -1, 0, 3/2$

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 3/2)$	$x \in (3/2, \infty)$
$x+1$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$2x-3$	$-$	$-$	$-$	$+$
$\frac{(x+1)(2x-3)}{x}$	$\boxed{-}$	$+$	$\boxed{-}$	$+$

To have non-zero denominator, we must have $x \neq 0$. Since we consider $<$ sign, the solution set is $(-\infty, -1) \cup (0, 3/2)$. ■

(b)

$$2 \leq \left|\frac{x-1}{x}\right| \leq 7$$

Solution: This is true when $2 \leq \left|\frac{x-1}{x}\right|$ and $\left|\frac{x-1}{x}\right| \leq 7$.

(i) Consider $2 \leq \left|\frac{x-1}{x}\right|$. This is equivalent to

$$\begin{aligned}
|2|^2 &\leq \left|\frac{x-1}{x}\right|^2 \\
4 &\leq \frac{(x-1)^2}{x^2} \qquad \text{since } \left|\frac{x-1}{x}\right|^2 = \frac{|x-1|^2}{|x|^2} = \frac{(x-1)^2}{x^2} \\
4x^2 &\leq (x-1)^2, \qquad \text{since } x^2 > 0 \\
4x^2 &\leq x^2 - 2x + 1 \\
3x^2 + 2x - 1 &\leq 0 \\
(x+1)(3x-1) &\leq 0.
\end{aligned}$$

By setting $(x+1)(3x-1) = 0$, we consider $x = -1$ and $x = 1/3$ to construct 3 subintervales.

	$x \in (-\infty, -1)$	$x \in (-1, 1/3)$	$x \in (1/3, \infty)$
$x+1$	$-$	$+$	$+$
$3x-1$	$-$	$-$	$+$
$(x+1)(3x-1)$	$+$	$\boxed{-}$	$+$

So, in this case, since $|x|$ is the denominator, $x \neq 0$ and the solution set is $\in [-1, 1/3] - \{0\}$ or $[-1, 0) \cup (0, 1/3]$

(ii) Consider $\left|\frac{x-1}{x}\right| \leq 7$ This is equivalent to

$$\begin{aligned}
\left|\frac{x-1}{x}\right|^2 &\leq |7|^2 \\
\frac{(x-1)^2}{x^2} &\leq 49 \qquad \text{since } \left|\frac{x-1}{x}\right|^2 = \frac{|x-1|^2}{|x|^2} = \frac{(x-1)^2}{x^2} \\
(x-1)^2 &\leq 49x^2, \qquad \text{since } x^2 > 0 \\
49x^2 &\geq x^2 - 2x + 1 \\
48x^2 + 2x - 1 &\geq 0 \\
(6x+1)(8x-1) &\geq 0.
\end{aligned}$$

By setting $(6x + 1)(8x - 1) = 0$, we consider $x = -1/6$ and $x = 1/8$ to construct 3 subintervales.

	$x \in (-\infty, -1/6)$	$x \in (-1/6, 1/8)$	$x \in (1/8, \infty)$
$(6x+1)$	$-$	$+$	$+$
$(8x-1)$	$-$	$-$	$+$
$(6x+1)(8x-1)$	$\boxed{+}$	$-$	$\boxed{+}$

So, in this case, since $|x|$ is the denominator, $x \neq 0$ and the solution set is $(-\infty, -1/6] \cup [1/8, \infty)$.

From (i) and (ii), the solution set is
$\{[-1, 0) \cup (0, 1/3]\} \cap \{(-\infty, -1/6] \cup [1/8, \infty)\} = [-1, -1/6] \cup [1/8, 1/3]$. ■

(c)

$$\frac{x^2 + |x| + 1}{x^7 + x^5 + x^2 + 1} \leq 0$$

Solution:
First notice that

$$x^7 + x^5 + x^2 + 1 = (x^7 + x^5) + (x^2 + 1) = x^5(x^2 + 1) + (x^2 + 1) = (x^5 + 1)(x^2 + 1).$$

Since $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$, we will consider two cases for x.

Case I: $x < 0$. Then $|x| = -x$ and

$$x^2 + |x| + 1 = x^2 - x + 1.$$

Notice that
$x^2 - x + 1 > 0$ for all $x \in \mathbb{R}$ $(b^2 - 4ac = 1 - 4(1)(1) < 0)$.

Case II: $x \geq 0$. Then $|x| = x$ and

$$x^2 + |x| + 1 = x^2 + x + 1.$$

Notice that
$x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$, $(b^2 - 4ac = 1 - 4(1)(1) < 0)$.
From cases (I) and (II), we have that the numerator

$$x^2 + |x| + 1 > 0$$

for all $x \in \mathbb{R}$. Not also that, since $x^2 \geq 0$ and $x^2 + 1 \geq 1 > 0$,

$$x^2 + 1 > 0, \qquad x \in \mathbb{R},$$

so we have that

$$\frac{x^2 + |x| + 1}{(x^5 + 1)(x^2 + 1)} < 0$$

only when

$$\frac{1}{(x^5 + 1)} < 0 \qquad \text{or } x^5 + 1 < 0$$

Note that $x^5 + 1 = 0$ when $x^5 = -1$ or $x = \sqrt[5]{-1} = -1$. So we consider the intervals $(-\infty, -1)$ and $(-1, \infty)$.

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
$x^5 + 1$	$\boxed{-}$	$+$

From above, the solution set is therefore $(-\infty, -1)$ ■

(d)

$$\frac{|x + 3| - 2}{5} + \frac{1}{|x - 1| + 1} \leq 1$$

(e)

$$\frac{|x - 1| - x^2 - 1}{5 - |x + 3|} \geq 0$$

Solution:

From $|x-1| = \begin{cases} -(x-1), & x - 1 < 0 \Leftrightarrow x < 1 \\ x - 1, & x - 1 \geq 0 \Leftrightarrow x \geq 1 \end{cases}$, and $|x+3| = \begin{cases} -(x+3), & x + 3 < 0 \Leftrightarrow x < -3 \\ x + 3, & x + 3 \geq 0 \Leftrightarrow x \geq -3 \end{cases}$,
we will consider 3 cases:

Case I: $x \in (-\infty, -3)$, Case II:$x \in [-3, 1)$, Case III: $x \in [1, \infty)$.

Case I: $x \in (-\infty, -3)$, $|x - 1| = -(x - 1)$, $|x + 3| = -(x + 3)$ and

$$\frac{x^2 + 1 - |x - 1|}{5 - |x + 3|} = \frac{x^2 + 1 + x - 1}{5 + x + 3} = \frac{x^2 + x}{x + 8} = \frac{x(x + 1)}{x + 8}.$$

So we will solve $\frac{x(x+1)}{x+8} \leq 0$. To obtain the subintervals, consider $x = -8, -1, 0$

	$x \in (-\infty, -8)$	$x \in (-8, -1)$	$x \in (-1, 0)$	$x \in (0, \infty)$
$x + 8$	$-$	$+$	$+$	$+$
$x + 1$	$-$	$-$	$+$	$+$
x	$-$	$-$	$-$	$+$
$\frac{x(x+1)}{x+8}$	$\boxed{-}$	$+$	$\boxed{-}$	$+$

To have non-zero denominator, we must have $x \neq -8$. Since we have "less than or equal" sign, then the solution set is $(-\infty,-8) \cup [-1,0]$. I.e., the solution set is $(-\infty,-3) \cap \{(-\infty,-8) \cup [-1,0]\} = \boxed{(-\infty,-8)}$.

Case II: $x \in [-3,1)$, $|x-1| = -(x-1)$, $|x+3| = x+3$ and

$$\frac{x^2+1-|x-1|}{5-|x+3|} = \frac{x^2+1+x-1}{5-x-3} = \frac{x^2+x}{-x+2} = -\frac{x(x+1)}{x-2}.$$

So we will solve $-\frac{x(x+1)}{x-2} \leq 0$ or $\frac{x(x+1)}{x-2} \geq 0$. To obtain the subintervals, consider $x = -1, 0, 2$

	$x \in (-\infty,-1)$	$x \in (-1,0)$	$x \in (-1,2)$	$x \in (2,\infty)$
$x+1$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$\frac{x(x+1)}{x-2}$	$-$	$\boxed{+}$	$-$	$\boxed{+}$

To have non-zero denominator, we must have $x \neq 2$. Since we have "less than or equal" sign, then $x \in [-1,0] \cup (2,\infty)$. I.e., the solution set is $[-3,1) \cap \{x \in [-1,0] \cup (2,\infty)\} = \boxed{[-1,0]}$.

Case III: $x \in [1,\infty)$, $|x-1| = x-1$, $|x+3| = x+3$ and

$$\frac{x^2+1-|x-1|}{5-|x+3|} = \frac{x^2+1-x+1}{5-x-3} = \frac{x^2-x+1}{-x+2} = -\frac{x^2-x+1}{x-2}.$$

So we want to solve

$$\frac{x^2-x+1}{x-2} \geq 0.$$

For $a=1, b=-1, c=1$, $b^2-4ac = 1-4 < 0$, so we cannot factor x^2-x+1. Since $a > 0$, then $x^2-x+1 > 0$ for all $x \in \mathbb{R}$. So, in order to have $\frac{x^2-x+1}{x-2} \geq 0$, we can consider instead $x-2 > 0$ or $x \in (2,\infty)$. That is, the solution set for this case is $[1,\infty) \cap (2,\infty) = \boxed{(2,\infty)}$.

From cases I, II, III, the solution set is $(-\infty,-8) \cup [-1,0] \cup (2,\infty)$. ■