Tutorial 2 Constraint Optimisation

1. Let $f(x,y) = 4x - 2x^2 - 2y^2$ defined over $S = \{(x,y) : x^2 + y^2 \le 4\}$
   (a) Find critical point(s) of f using first derivative test.
   (b) Apply the extreme value theorem to find absolute maxima and minima of f over S.
   [Ans: Critical point is (1,0). Max. $f(1,0) = 2$. Min. $f(-2,0) = -16$.]

2. Use the substitution method, solve the following constrained optimisation problem. Determine the optimum values of the choice variables, and use the second-order condition to verify that they are relative minima or maxima.

   (a) $y = x^2 + 2xz + 4z^2$ subject to $x + z = 8$ [Ans: $(x,z) = (8,0)$ relative minimum]

   (b) $y = 10x + 40z$ subject to $x^{\frac{1}{2}} z^{\frac{1}{2}} = 2$ [Ans: $(x,z) = (4,1)$ relative minimum]

   (c) $y = \ln 2x + 2\ln z$ subject to $z = 16 - 4x$ [Ans: $(x,z) = \left(\frac{4}{3}, \frac{32}{3}\right)$ relative maximum]

3. Use the substitution method to find the optimum points for the multivariable function

$$U = 2x^2 + 5xy - y^2 - 3xz$$

subject to

$$x + y + z = 14.$$

Use the second-order derivative test to confirm whether the extreme value represents a relative maximum, relative minimum or saddle point. Ans: $(x,y,z) = (1,4,9)$ saddle point]

4. (a) Find the optimum values of

$$f(x,y) = 3x^2 + xy + 2y^2$$

   subject to the constraint

$$x + y = 200.$$

   [Ans: $f(75,125) = 57{,}500$]

   (b) A firm producing 2 products, Product X with quantity x and Product Y with quantity y. The total cost is

$$C = 3x^2 + xy + 2y^2.$$

   The capacity of production is

$$x + y \le 200.$$

   Find the optimum total cost if the firm want to produce something!
   [Ans: $(x,y) = (0,0)$ is not considered because the firm want to produce something. The optimum total cost is 57,500 when the firm produces 75 of X and 125 of Y.]

5. Use the Lagrange multiplier method to determine the optimum values for each constrained optimization problem and solve for the value of the corresponding Lagrange multiplier.
   (a) $U = x^2 + 2x + 3z^2 - 6z + xz$ subject to $2x + 2z = 32$
   [Ans: $(x,z) = (12,4), \lambda = 15$]

   (b) $U = \frac{1}{2}a^2 + 4b^2 - 4a + 8c^2$ subject to $\frac{1}{2}a + 3b + c = 25$
   [Ans: $(a,b,c) = (12,6,1), \lambda = 16$]

   (c) $U = \ln(x + y)$ subject to $xy = 16$ [Ans: $(x,y) = (4,4), \lambda = \frac{1}{32}$]

6. Use the Lagrange multiplier method to determine the extreme values of the function

$$Z = x^2 + 2xy + w^2$$

subject to the two constraints

$$2x + y + 3w = 24 \text{ and } x + w = 8$$

[Ans: $(w, x, z) = (6,2,2); \lambda_1 = 4; \lambda_2 = 0$]

7. Maximise $f(x, y) = 2x + y - xy$ subject to $x + 2y \le 10$. Find the maximum value of $f(x, y)$. Be sure to check that these optimum solutions satisfy all the complementary slackness conditions.

[Ans: Critical point is (1,2) but $\left(\frac{7}{2}, \frac{13}{4}\right)$ is not because $\mu < 0$. $f(x, y)$ is maximized when $x = 1, y = 2$ and $f(1,2) = 2$]

8. The Cobb-Douglass production function of a new product is given by $N(x, y) = 2x^{0.5} y^{0.5}$ where x is the number of units of labour and y is the number of unit of capital required to produce $N(x,y)$ units of the product. Each unit of labour costs \$4 and each unit of capital costs \$12. If \$72 has been budgeted for the production of this product ($4x + 12y = 72$). In addition, the number of units of labour is limited to 16 due to the limitation of space ($x \le 16$). How should this amount be allocated between labour and capital in order to maximise production? [Ans: x = 9 and y = 3]

9. Maximise $f(x, y) = xy + 2y$ subjected to $2x + y = 30$ and $y \le 15$.

[Ans: $f(7.5,15) = 142.5$ is maximised., $f(6.5,17) = 144.5$ is not since $y > 15$]

10. Find the maximum value of the function

$$h(x, y) = 2x^2 - y^2$$

subject to the constraint that

$$x^2 + y^2 = 1$$

and the constraint that x and y are nonnegative. Be sure to check that these optimum solutions satisfy all the complementary slackness conditions.

Hint: We can use Extreme value theorem or Lagrange multiplier method to solve.
[Ans: $L = 2x^2 - y^2 - \lambda(x^2 + y^2 - 1) + \mu_1 x + \mu_2 y$, there are 9 equations to solve for 6 unknowns. The optimum solution is x = 1, y = 0 when $\mu_1 = 0$.]

11. Find critical point(s) of $f(x, y) = 6x + 4e^{-x} + e^{y} - 0.5e^{2y}$ subject to $x + y = 0$
(a) Use the substitution method (5 marks)
(b) Use the Lagrange multiplier method (7 marks)

[Ans: $(x, y) = (-\ln 3, \ln 3)$ and $(x, y) = (-\ln 2, \ln 2)$]

Additional:

12. Solve optimization problem $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x^2 + xy + y^2 = 3$

[Ans. (1,1),(-1,1),($\sqrt{3}$,- $\sqrt{3}$),(- $\sqrt{3}$, $\sqrt{3}$)]

13. Find maximum value of $f(x,y) = 10x^{1/2}y^{1/3}$ subject to $2x + 4y = m$

14. Find maximum value of $f(x,y,z) = x^2 + y^2 + z^2$ subject to $x + 2y + z = 30$ and $2x - y - 3z = 10$

[Ans. f(10,10,0) = 200]

15. Find critical point(s) of $f(x,y,z,w) = 3x^2 + y^2 + 2z^2 - 5w^2$ subject to $x + 6y + 3z + 2w = 4$

16. Find maximum value of $f(x,y) = x^2 + 2y$ subject to $x^2 + y^2 \le 5, \quad -y \le 0$

[Ans. f(2,1) = 6, f(-2,1)=6]

17. Find maximum value of $f(x,y) = x + ay$ subject to $x^2 + y^2 \le 1, \quad x + y \ge 0$ where a is a constant