Macroeconomics

Lecture 3

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Bellman's Equations

- Define the (T+1)-period value function

$$W_{T+1}(x_0) = \max_{c_0,\ldots,c_T} \left[U_0(x_0,c_0) + U_1(x_1,c_1) + \ldots + U_T(x_T,c_T) \right] \qquad (1)$$

$$\textit{Subject to} \qquad x_{t+1} = g_t(x_t,c_t), \qquad t = 0,1,\ldots,; \quad x_0 \ \textit{given.}$$

Problem (1) is the same as

$$W_{T+1}(x_0) = \max_{c_0}\left[U_0(x_0,c_0) + \max_{c_1}\left[U_1(x_1,c_1) + \ldots . \max_{c_T}\left[U_T(x_T,c_T) \right]\right]\right] \ (2)$$

$$\textit{Subject to} \quad x_{t+1} = g_t(x_t,c_t), \qquad x_0 \ \textit{given.}$$

(1) Solving for $c_T=h_T(x_T)$ to optimize $W_1(x_T)$,

(2) Solving for $c_{T-1}=h_{T-1}(x_{T-1})$ to optimize $W_2(x_{T-1})$,

Continue to repeat this process until t=0.

Self-enforcing property

- The optimal policies $c_t=h_t(x_t)$, t=0,1,2…,T, from problem (1) will have the same values of

  $c_s=h_s(x_s)$, t=s,s+1,s+2…,T,

  as those optimal policies obtained from the following problem: for s>0

$$\max_{c_s,\dots,c_T}\left[U_s\left(x_s,c_s\right)+U_{s+1}\left(x_{s+1},c_{s+1}\right)+\dots+U_T\left(x_T,c_T\right)\right] \qquad (3)$$

$$\textit{Subject to} \qquad x_{t+1}=g_t\left(x_t,c_t\right), \qquad t=s,s+1,\dots; \quad x_s \ \textit{given.}$$

- Then, these optimal policies are said to be “time consistent”.

Policy function that does not vary over time

- Assume that: $U_t(x_t,c_t) = \beta^t.U(x_t,c_t)$, $0<\beta<1$
- $g_t(x_t,c_t) = g(x_t,c_t)$, (4)
- Bellman's equation becomes (for j=1,2,..T)

$$W_{j+1}\left(x_{T-j}\right) = \max_{c_{T-j}}\left[\beta^{T-j}U\left(x_{T-j},c_{T-j}\right) + W_j\left(x_{T-j+1}\right)\right]$$

$$\beta^{j-T}W_{j+1}\left(x_{T-j}\right) = \max_{c_{T-j}}\left[U\left(x_{T-j},c_{T-j}\right) + \beta.\beta^{j-T-1}W_j\left(x_{T-j+1}\right)\right]$$

$$V_{j+1}\left(x_{T-j}\right) = \max_{c_{T-j}}\left[U\left(x_{T-j},c_{T-j}\right) + \beta.V_j\left(x_{T-j+1}\right)\right] \qquad (5)$$

- where

$$V_{j+1}\left(x_{T-j}\right) = \beta^{j-T}W_{j+1}\left(x_{T-j}\right)$$

$$If \quad j=T, \ then \qquad V_{T+1}\left(x_0\right) = W_{T+1}\left(x_0\right)$$

Policy function that does not vary over time

The Bellman's equation becomes, (for j=1,2,….,T)

$$V_{j+1}\left(x_{T-j}\right) = \max_{c_{T-j}}\left[U\left(x_{T-j},c_{T-j}\right) + \beta V_j\left(x_{T-j+1}\right)\right] \qquad (6)$$

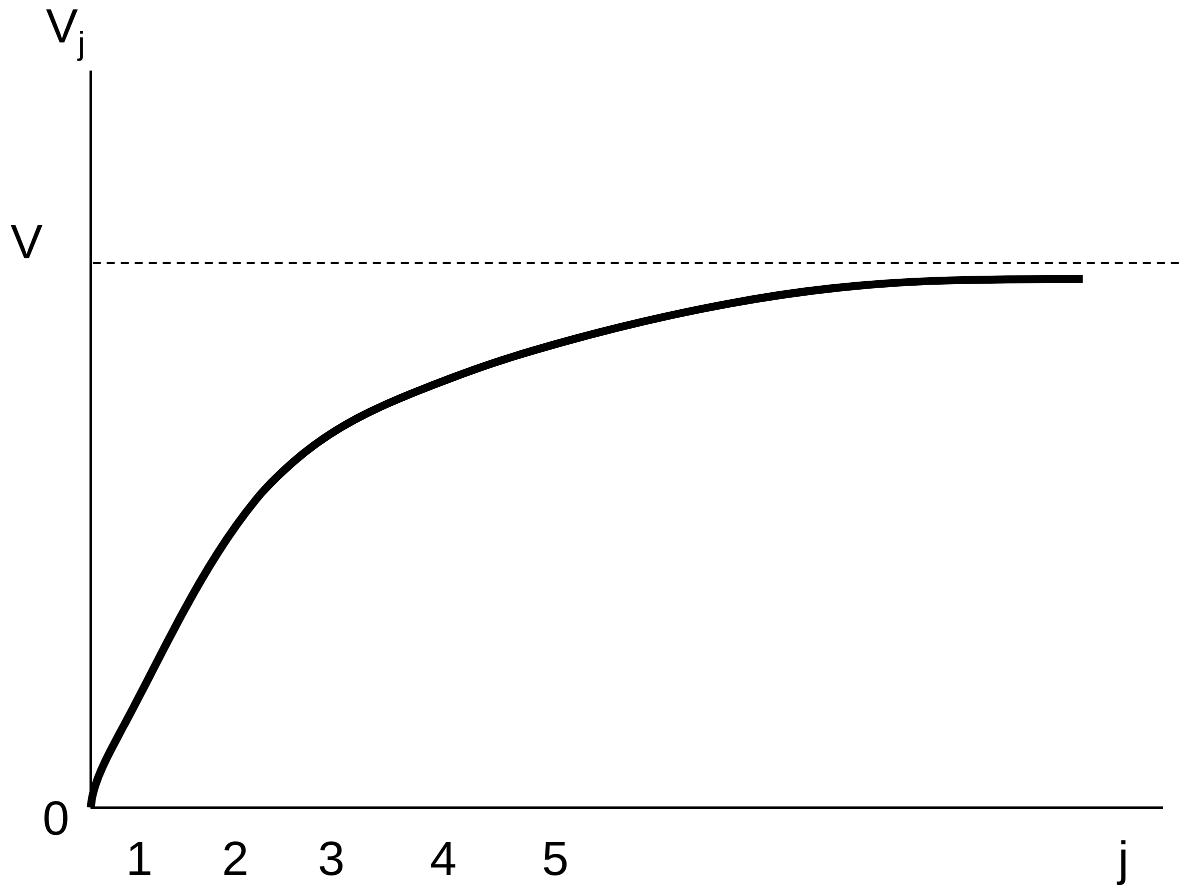
$$Subject\ \ to \qquad x_{T-j+1} = g\left(x_{T-j},c_{T-j}\right), \quad x_{T-j} \ \ given$$

- In the case that $V=\lim_{j\rightarrow\infty} V_j$, then

$$V\left(x\right) = \max_{c}\left[U\left(x,c\right) + \beta V\left(\tilde{x}\right)\right] \qquad (7)$$

$$Subject\ to \qquad \tilde{x} = g\left(x,c\right), \ \ x \ \ given.$$

Policy function that does not vary over time



Discounted Dynamic Programming Problem

- The limiting value function V that solves (7) is the optimal value function for the infinite horizon problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, c_t) \qquad (8)$$

$$\textit{Subject to} \quad x_{t+1} = g(x_t, c_t), \quad x_0 \ \textit{given}$$

- There is a unique and time-invariant optimal policy
- $c_t = h(x_t)$, t=0,1,2…,T
- Where h(.) is chosen to maximize the right hand side of (7), and

Discounted Dynamic Programming Problem

the limiting value function V is differentiable with respect to x. (This is the same as the marginal condition (1.4d) in Lecture #1). Hence,

$$V'(x) = \frac{\partial U[x, h(x)]}{\partial x} + \frac{\partial U[x, h(x)]}{\partial h(x)} \cdot \frac{\partial h(x)}{\partial x} + \beta \frac{\partial g[x, h(x)]}{\partial x} V'(g[x, h(x)]) \qquad (9)$$