Lecture 4
Optimal Risky Portfolios

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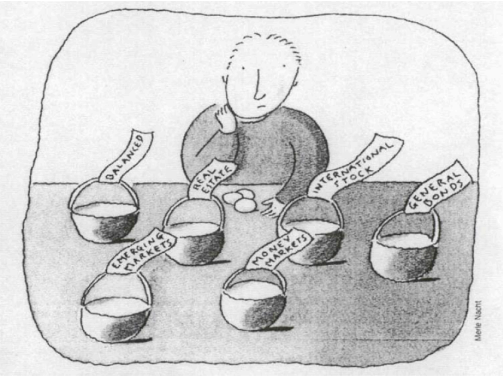
FN 312 – INVESTMENTS Fall 2016

Introduction

- Having determined the appropriate exposure to risk, the investor's next task is to build the risky portfolio
- This selection will be made from the whole universe of risky assets
- How much of your wealth should you invest in each security?



Mean Variance Analysis

- Up until the mean-variance analysis of Markowitz became known, an investment advisor might have given you the following advice:
  - If you are young you should put money into a couple of good growth stocks, maybe even into a few small stocks. Now is the time to take risk
  - If you are close to retirement, you should put all of your money into bond and safe stocks, and nothing into risky stocks. Don't take risks with your portfolio at this stage in your life!

Mean-Variance Analysis: Preview

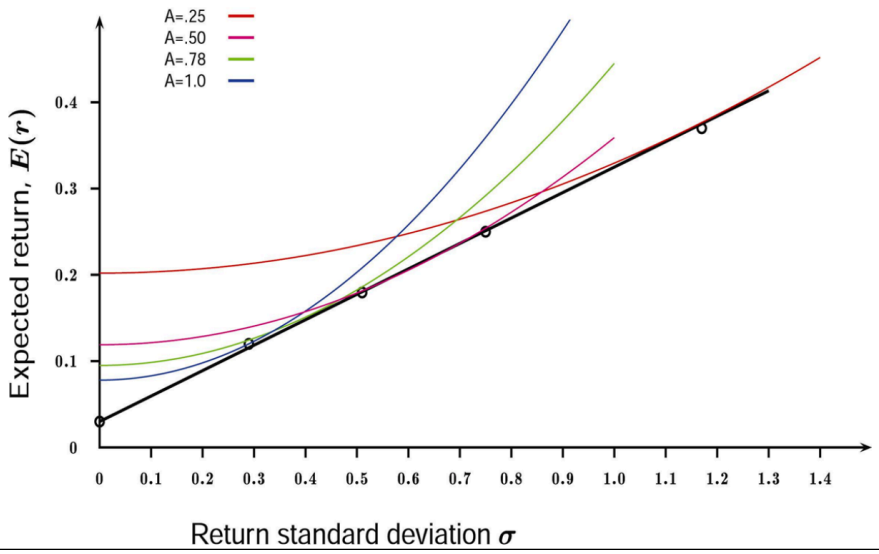
- We will show that the optimal portfolio of risky assets is exactly the same for everyone, no matter what their tolerance for risk
  - Investors should control the risk of their portfolio not by reallocating among risky assets, but through the split between risky and risk-free assets
  - The portfolio of risky assets should contain a large number of assets - it should be a well diversified portfolio

Modern Portfolio Theory

- Harry Markowitz
- Adjunct Professor of Finance
- UC San Diego

- An economist who devised the modern portfolio theory: 'Portfolio Selection', *The Journal of Finance*, 1952 (Published while a graduate student at the University of Chicago at 25 years old)
- Best known for his work in modern portfolio theory, emphasizing the importance of diversification
- Markowitz won the Nobel Prize in 1990, while a professor at CUNY
- A Markowitz Efficient Portfolio is one where no added diversification can lower the portfolio's risk for a given expected return

Recap: Capital Allocation



Optimal Risky Portfolio

Start simple: Two assets

- The expected return for the portfolio is
- The variance of the portfolio is:
- The variance of the portfolio depends on the correlation between the two securities

Optimal Risky Portfolio

- What should be the combination of risky securities in your portfolio?
- Plot out possible set of expected returns and standard deviations for different combinations of the assets
- *Minimum Variance Frontier* is the set of portfolios with the lowest variance for a given expected return

Two Risky Asset Example

Asset	$E(\tilde{r})$	σ
A	25%	75%
B	10%	25%

- How does correlation affect the risk? Derive the minimum variance frontier under 3 different assumptions
  1. $\rho_{AB} = 1$
  2. $\rho_{AB} = -1$
  3. $\rho_{AB} = 0$

Case $\rho_{AB} = 1$

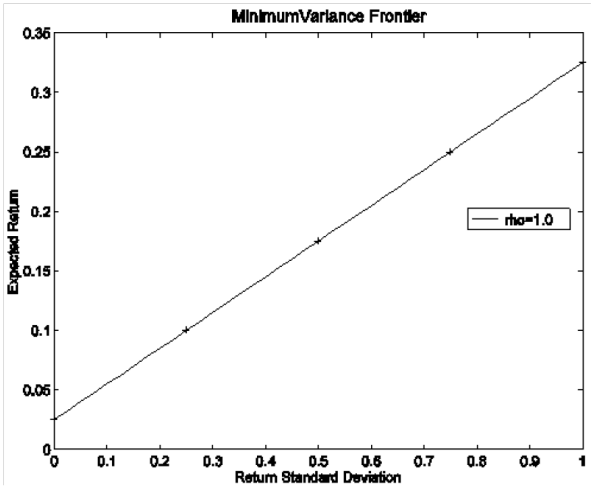
- Plug these numbers into these two equations:

$$\begin{aligned} E(\tilde{r}_p) &= 0.25w + 0.10 \cdot (1-w) \\ &= 0.15w + 0.10 \\ \sigma_p &= 0.75w + 0.25 \cdot (1-w) \\ &= 0.50w + 0.25 \end{aligned}$$

- In *Excel*, we can build a table with various possible w's:

w	$E(\tilde{r}_p)$	σ_p
-.5	2.5%	0.0%
0	10%	25%
0.5	17.5%	50.0%
1	25.0%	75.0%
1.5	32.5%	100.0%

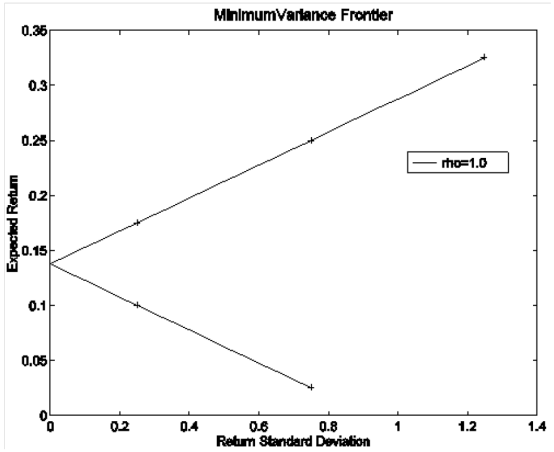
No diversification benefit



Case $\rho_{AB} = -1$

- When $\rho = -1$ we can again simplify the variance equation:

- Again, if we create a table of the expected returns and variances for different weights and plot these, we get: (here for $-0.5 \leq w \leq 1.5$):

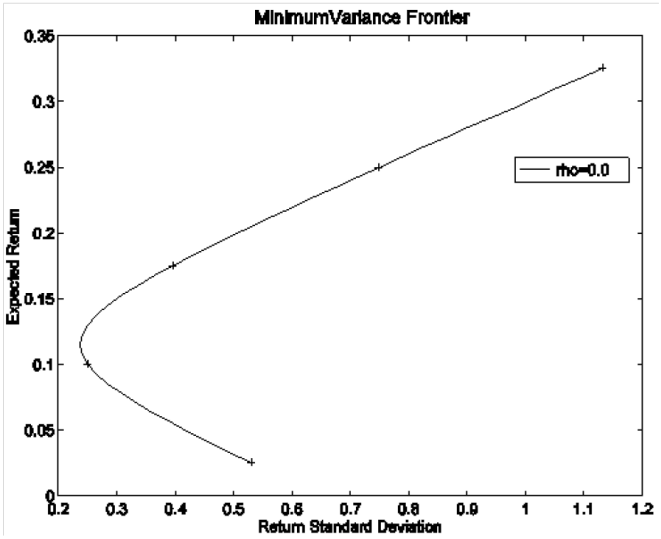


Perfect Hedge

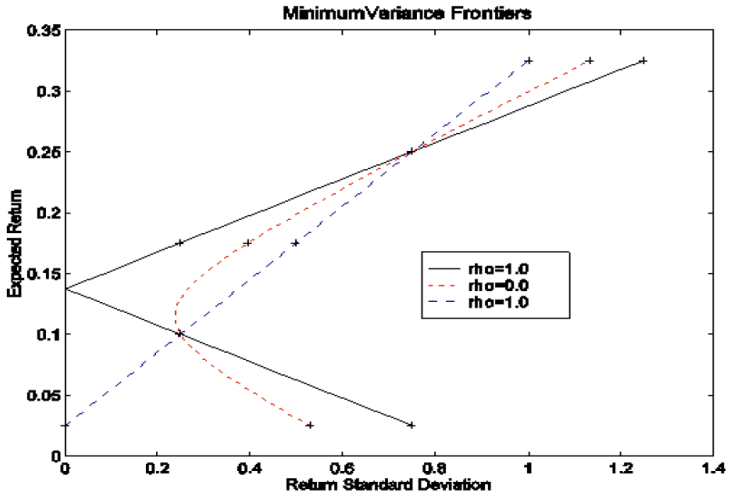
- Perfect Hedge is a hedge that gives a portfolio with zero risk
- It is possible to find a perfect hedge with these 2 securities

- We have created a 'synthetic' risk-free security!

Two Risky assets, $\rho_{AB} = 0$



All cases together



Correlation and diversification

- Diversification benefit

- Hedged Assets

Many risky assets …

- Three risky assets

- N asset portfolio

Shape of the Portfolio Possibilities Curve

What if a risk-free asset is available?

- We have covered the capital allocation problem between a risk-free asset and a risky asset
- Recall that the capital allocation line is the straight line through the risk-free asset and the risky asset
- Adding the risk-free asset generates a number of CAL choices
  - Which CAL to choose?

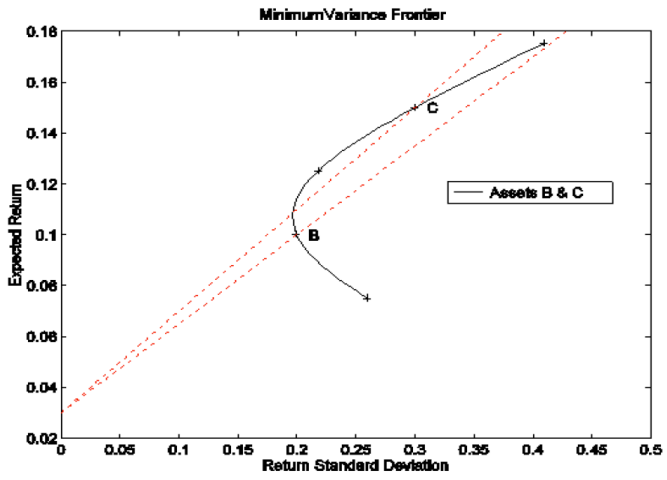
CAL with two risky assets

- Assume we can only trade in risk-free asset with 3% return and risk assets B and C where

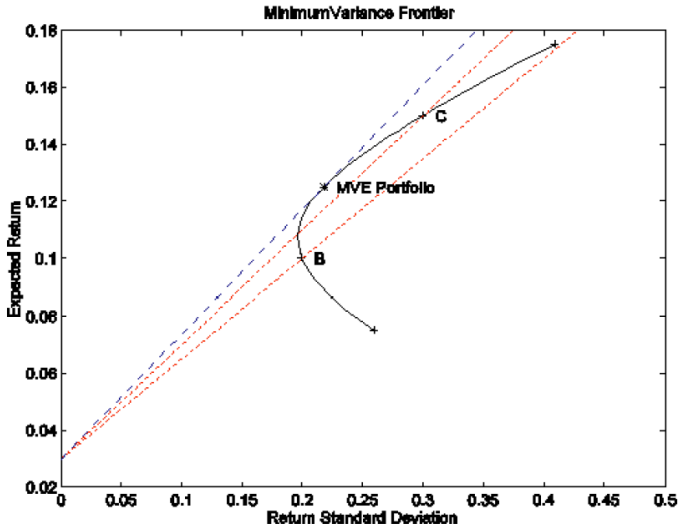
Asset	$E(r)$	σ
B	10%	20%
C	15%	30%

  with correlation coefficient 0.5
- We can compute the minimum variance frontier created by combinations of B and C
  - The risk-free asset plus B, or the risk-free asset plus C gives us two possible CALs

CAL with two risky assets



The Mean-Variance Efficient (MVE) Portfolio



Finding the MVE Portfolio (Algebra)

Optimal Risky Portfolio

- Adding the risk free asset generates a number of CAL choices
- The feasible set of portfolios becomes more attractive
- We can identify an optimal risky portfolio which dominates all other risky portfolios (irrespective of risk preferences)
- The optimal (tangency/MVE) portfolio has the highest Sharpe ratio among all feasible portfolios (highest slope)

Example: Optimal Risky Portfolio

- Investing $100,000, risk-free = 0.5%
- Stock 1 has expected return 8%, stdev 12%
- Stock 2 has expected return 13%, stdev 20%
- The correlation between the two stock returns is zero.

Optimal Complete Portfolio

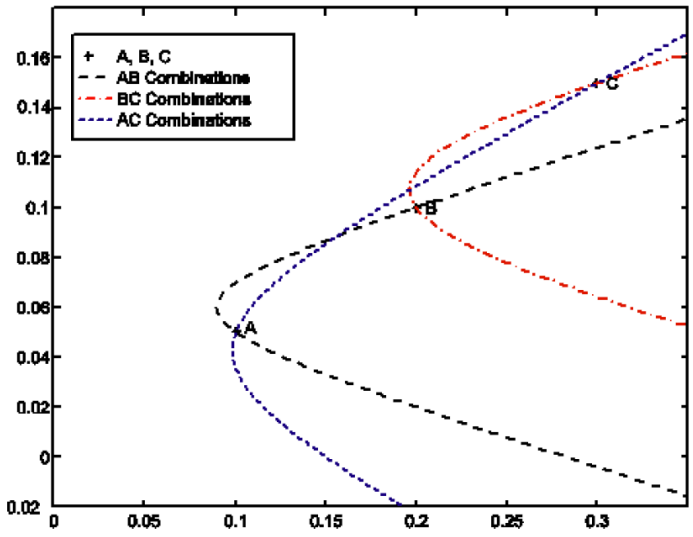
- Investing $100,000, A=8, risk-free = 0.5%

Optimal Portfolio with Many Risky Assets

Asset	$E(r)$	σ
A	5%	10%
B	10%	20%
C	15%	30%

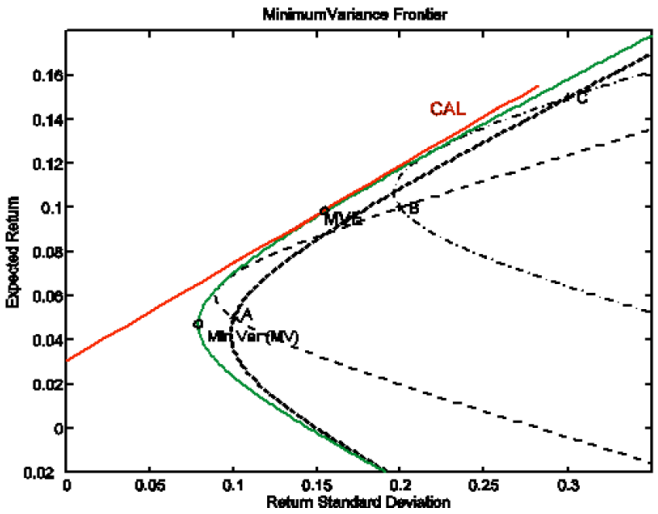
Correlations			
Assets	A	B	C
A	1.0	0.0	0.5
B	0.0	1.0	0.5
C	0.5	0.5	1.0

- What does the minimum variance frontier look like now?

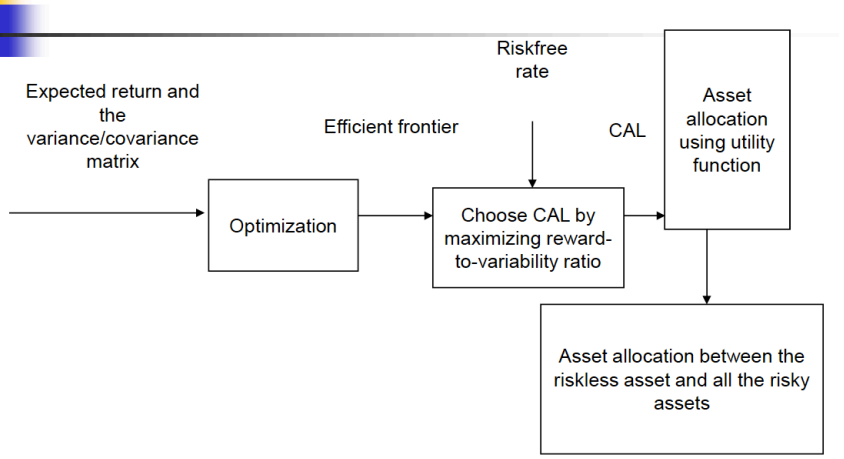


- We get the possible combinations. Can we do better?

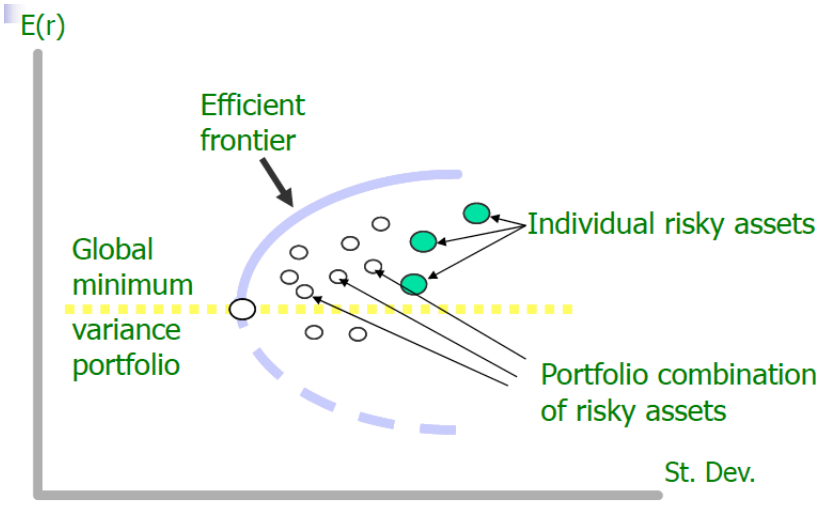
Portfolios of Portfolios



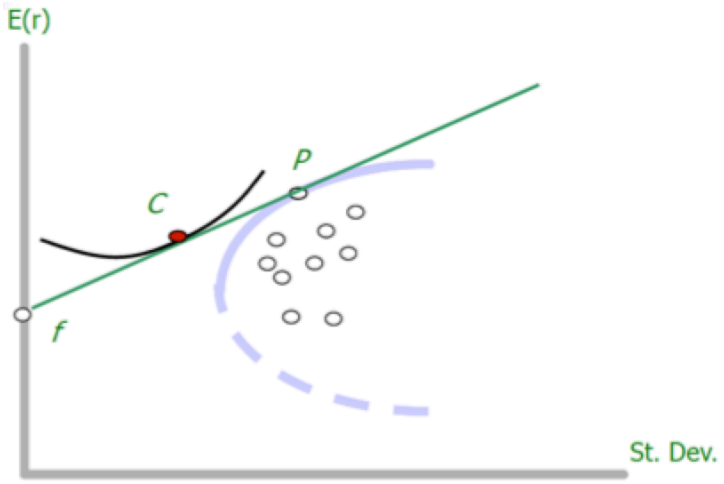
Markowitz Portfolio Selection Model



The Case of N Risky Assets



The Case of N Risky Assets (cont.)



Optimal Portfolio Selection

1) Estimate expected return, variances-covariances matrix
2) The weights are determined by quadratic programming optimization with constraints based on either:
   - For given expected return, find the minimum variance
   - For given variance, find the maximum expected return

Note: You can do this by using Solver in Excel

$$\underset{w_i}{\text{Min}} \ \sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j>i}^{n} w_i w_j \sigma_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i = 1$$

$$\sum_{i=1}^{n} w_i \mu_i = \bar{\mu}_p$$

$$w_i \geq 0$$

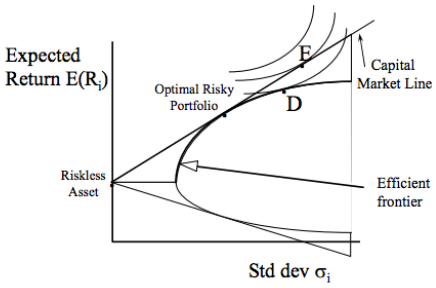
$$|w_i| < k$$

4) Trace out the minimum variance frontier by repeating the optimization for all levels of expected return

5) Graph the efficient frontier

6) With a risk-free asset, find CAL with highest Sharpe ratio, with the point of tangency with efficient frontier being the optimal risky portfolio

6) Find the optimal complete portfolio for a specific investor using his/her utility function



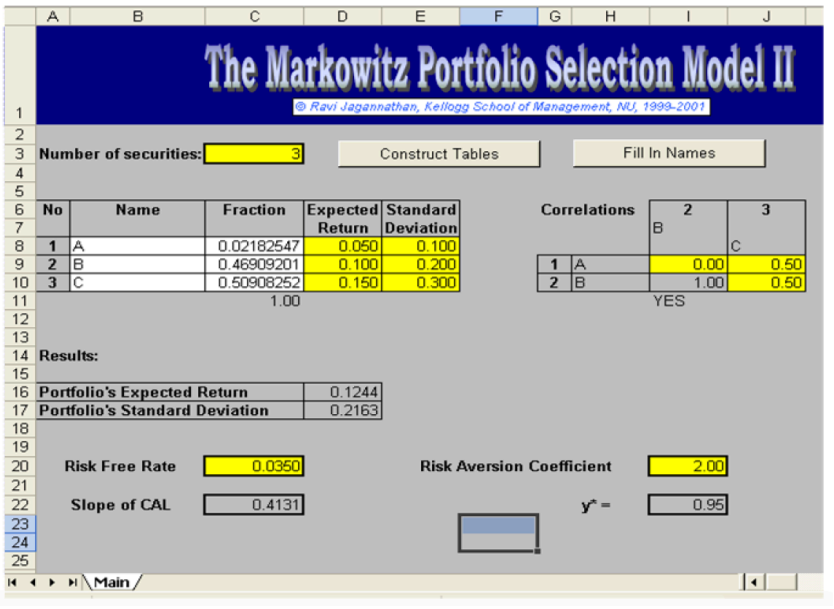
Example

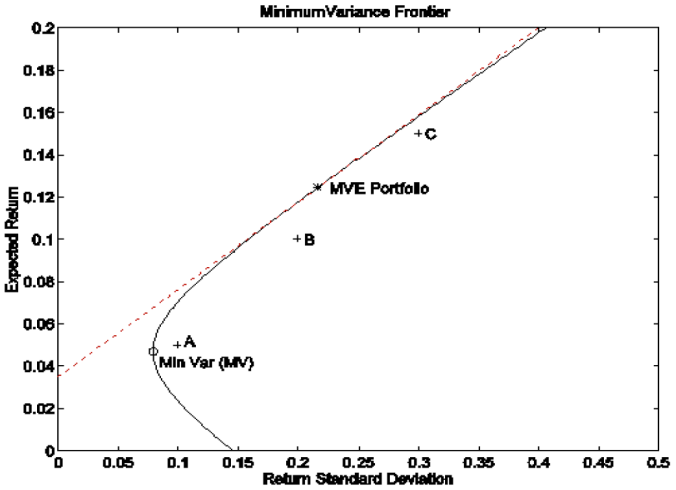
- Consider three risky assets, A, B, C with risk free return at 3.5%

Asset	$E(r)$	σ
A	5%	10%
B	10%	20%
C	15%	30%

Correlations			
Assets	A	B	C
A	1.0	0.0	0.5
B	0.0	1.0	0.5
C	0.5	0.5	1.0

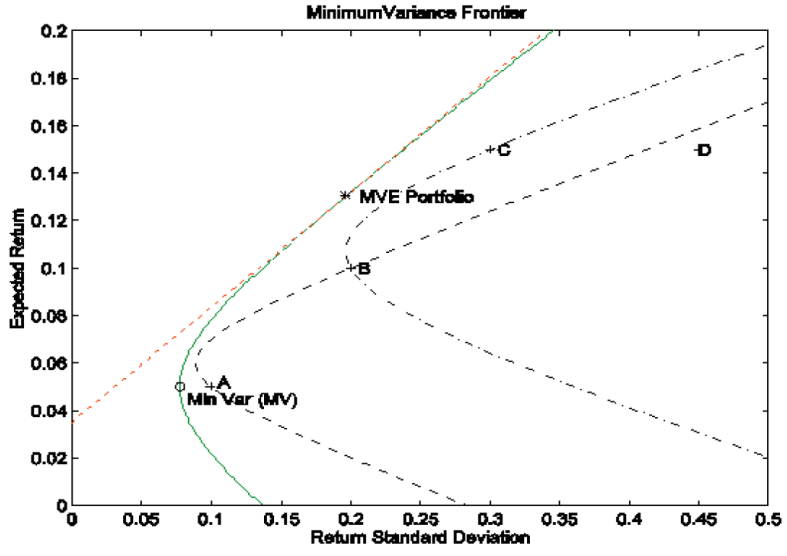
- Refer to Markowitz.xls and tutorial on moodle





$$\mathbf{w}_{MVE} = \begin{bmatrix} 0.0218 \\ 0.4619 \\ 0.5091 \end{bmatrix} \qquad \begin{aligned} \bar{r}_{MVE} &= 0.1244 \\ \sigma_{MVE} &= 0.2163 \\ SR_{MVE} &= 0.4131 \end{aligned}$$

- Let's add a fourth security, say, D with $E(r_D) = 15\%$, $\sigma_D = 45\%$.
- Assume that it is a zero correlation with all of the other securities.
- Will anyone want to hold this security?



- The optimal allocation looks like this

$$\mathbf{w}_{MVE} = \begin{bmatrix} 0.0168 \\ 0.3616 \\ 0.3924 \\ 0.2292 \end{bmatrix}$$

$$\bar{r}_{MVE} = 0.1302$$
$$\sigma_{MVE} = 0.1961$$
$$SR_{MVE} = 0.4858$$

- What explains this?
  - ↪ D is apparently strictly dominated by C
  - ↪ D is uncorrelated with A,B,C
  - ↪ How can D contribute to the overall portfolio?
  - ↪ Wouldn't increasing the share of C by 23% dominate the allocation above?

Passive Strategy is Efficient

- Basic message: your risk/return tradeoff is improved by holding many assets with less than perfect correlation
- The optimal risky portfolio is the same for every investor, and is the market portfolio
- No need for stock selection
- Investors need only to adjust the mix of risk-free asset and the market portfolio based on risk aversion

Far from everyone agrees…

"To suppose that safety-first consists in having a small gamble in a large number of different companies where I have no information to reach good judgment, as compared with a substantial stake in a company where one's information is adequate, strikes me as a travesty of investment policy"	"Diversification is an admission of not knowing what to do and striking an average"	"…and is the business understandable? Despite high regard for Microsoft, Mr. Buffet avoids its stock because the field puzzles him. Ignorance, he says, increases danger. This belief is a departure from the common wisdom of stock diversification. Owning many different stocks – good, bad and mediocre – depresses the returns a more selective portfolio would achieve, he believes, and makes it impossible to understand all that you own. Thus in 1987 his $2 billion portfolio had just three companies. Today nearly $15 billion is spread among just 10."
- J.M. Keynes 1939	**- G. Loeb, 1935**	**- W. Buffet, 1996**

Why?

Reconsider the assumptions for mean-variance analysis:

- Separation Theorem
- No market friction
- No heterogeneity in investors
- Static expected returns and variances
- Risk averse investor

Final notes on diversification

1. Start with our equation for variance:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j cov(\tilde{r}_i, \tilde{r}_j)$$

2. Then make the simplifying assumption that $w_i = 1/N$ for all assets:

$$\sigma_p^2 = \left(\frac{1}{N^2}\right) \sum_{i=1}^{N} \sigma_i^2 + \sum_{i=1}^{N} \left(\frac{1}{N^2}\right) \sum_{\substack{j=1 \\ i \neq j}}^{N} cov(\tilde{r}_i, \tilde{r}_j)$$

3. The average variance and covariance of the securities are:

$$\overline{\sigma^2} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \sigma_i^2 \qquad \overline{cov} = \frac{1}{N(N-1)} \sum_{\substack{j=1 \\ i \neq j}}^{N} cov(\tilde{r}_i, \tilde{r}_j)$$

1. Plugging these into our equation gives:

$$\sigma_p^2 = \left(\frac{1}{N}\right) \overline{\sigma^2} + \left(\frac{N-1}{N}\right) \overline{cov}$$

2. What happens as N becomes large?

$$\left(\frac{1}{N}\right) \to 0 \text{ and } \left(\frac{N-1}{N}\right) \to 1$$

3. *Only the average covariance matters for large portfolios.*
4. If the average covariance is zero, then the portfolio variance is close to zero for large portfolios.

- The component of risk that can be diversified away we call the *diversifiable* or *non-systematic* risk.
- Empirical Facts
  - ↪ The average (annual) return standard deviation is 49%
  - ↪ The average (annual) covariance between stocks is 0.037, and the average correlation is about 39%.
- Since the average covariance is positive, even a very large portfolio of stocks will be risky. We call the risk that cannot be diversified away the *systematic risk.*

Conclusion

- We have developed mean-variance portfolio analysis
  - We assume that all that matters to investors is the average return and the return variance of their portfolio
  - This is appropriate if returns are normally distributed
- Couple of lessons
  - Hold the same portfolio of risky assets no matter what your tolerance for risk
    - If you want less risk, combine the portfolio with investment in the risk-free asset
    - If you want more risk, buy the portfolio on margin
  - In large portfolios, covariance is important, not variance

Readings

- Readings:
  - Campbell, J.Y., M. Lettau, B.G. Malkiel, Y. Xu, 2001, "Have Individual Stocks Become More Volatile? An Empirical Exploration of Idiosyncratic Risk", Journal of Finance 56, 1-43.
  - Markowitz, H., 1952, "Portfolio Selection", The Journal of Finance 7, No. 1, 77-91.

(Files are posted on moodle)

Homework

BKM Ch 7

- Solve #4-10, then repeat the analysis with Excel Solver to check your answer (see tutorial on how to use Excel Solver in lecture notes folder on moodle, and also refer to Appendix A of book)
- Solve #17-19