4.2 Solving Inequalities without Absolute Values

Solving inequality is to find the solution of the variable that appears in the inequality. The solution is the real number which satisfies the inequality. The inequality can have more than one solution which we call the set of the solutions as the ***solution set of the inequality***.

4.2.1 Solving the linear inequality

Linear inequality is the inequality in the form $ax+b<cx+d$ (or $ax+b\leq cx+d$) where *a*, *b*, *c*, and *d* are constants and *a* and *b* cannot be zero at the same time. We consider linear inequality in two cases:

1) Linear inequality in the form of $ax+b<cx+d$ (or $ax+b\leq cx+d$) where *a*, *b*, *c*, and *d* are constants and $a\neq c$. We solve the linear inequality by rearrange the inequality to have the constant on one side and the coefficient of the variable is 1.

Example 4.2.1 Find the solution set of the following inequalities.

(a) $3x-1\geq x+5$ (b) $2-x<2x-4$

(c) $\frac{x}{2}+1\leq x-5$ (d) $x+1>5x-11$

2) Linear inequality in the form of $ax+b<ax+d$ (or $ax+b\le ax+d$) where *a*, *b*, and *c* are constants and $a\ne 0$. We solve this type of linear inequality by considering the value of *b* and *c* if whether they satisfy the inequality or not. That is if $b<c$ (or $b\le c$) then the solution set of the inequality $ax+b<ax+d$ (or $ax+b\le cx+d$) is the set of real number ($\mathbb{R}$) because the inequality is true for all values of x. If $b\ge c$ (or $b>c$) then the solution set of the inequality $ax+b<ax+d$ (or $ax+b\le cx+d$) is an empty set ($\varnothing$) because there is no value of x that satisfy the inequality.

Example 4.2.2 Find the solution set of the following inequalities.

(a) $x+4\le x-3$

(b) $2-3x>-2-3x$

(c) $4x-1<4x-1$

(d) $3+\frac{2x}{3}\ge\frac{2x}{3}-2$

4.2.2 Solving the polynomial inequalities

Polynomial function is the function of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ where $a_0, a_1, a_2, ..., a_n$ are constants and n is a positive integer or zero and n is the degree of the polynomial. The procedures of solving the polynomial inequalities are as following:

1) Rearrange the inequality to have zero on one side.
2) Write the polynomial in the form of linear factors. If the polynomial cannot be written in the form of linear factors then consider whether the polynomial satisfy the inequality or not. If it does then we conclude that the solution set is a set of real numbers otherwise we conclude that the solution set is an empty set.
3) Find the value of the variable which makes the linear factors equal to zero. These values are called the ***critical values.***
4) Find the value of the variable that satisfies the inequality by considering the real number on the intervals near the critical values.

Note: The polynomial with degree two, $ax^2 + bx + c$, can be factored when $b^2 - 4ac \geq 0$.

Example 4.2.3 Find the solution set of the inequality $2x^2 - 5x \leq 3$.

Example 4.2.4: Solve the inequality $x^2 + 8 > 6x$.

Example 4.2.5: Find the solution set of the inequality $x^3 + x^2 \geq 2x$

Example 4.2.6: Find the solution set of the inequality $\left(x^2+5x+4\right)\left(2x^2-x+3\right)<0$.

Example 4.2.7: Find the solution set of the inequality $\left(-2x^2-x-4\right)\left(x^2+2x+1\right)\geq 0$.

4.2.3 Solving the inequalities involving rational functions

Rational function is the function in the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$. The method of solving the inequalities involving rational functions is similar to the method of solving polynomial inequalities.

Example 4.2.3.1: Solve the inequality $\frac{1}{x+4} < \frac{1}{x}$.

Example 4.2.3.2: Find the solution set of the inequality $\dfrac{x^3 - x^2}{x+1} + 2x > 0$.

Example 4.2.3.3: Solve the inequality $\dfrac{x^2}{-x^2 + x - 4} \le -1$.