Solution: Assignment 1

1. Determine whether the statement forms are logically equivalent. In each case, construct a truth table to justify your answer.

   (a) $p \to q \vee r$, $\quad p \wedge \sim q \to r$, $\quad p \wedge \sim r \to q$.

   (b) $(p \wedge q) \vee r$, $\quad p \wedge (q \vee r)$

   (c) $p \to (q \to r)$, $\quad (p \wedge q) \to r$

   <u>Answer:</u>
   (a) Using the order of connective operations gives $p \to (q \vee r)$, $\quad (p \wedge \sim q) \to r$, $\quad (p \wedge \sim r) \to q$.
   Truth table:

p	q	r	$\sim q$	$\sim r$	$q \vee r$	$p \wedge \sim q$	$p \wedge \sim r$	$p \to (q \vee r)$	$(p \wedge \sim q) \to r$	$(p \wedge \sim r) \to q$
T	T	T	F	F	T	F	F	T	T	T
T	T	F	F	T	T	F	T	T	T	T
T	F	T	T	F	T	T	F	T	T	T
T	F	F	T	T	F	T	T	F	F	F
F	T	T	F	F	T	F	F	T	T	T
F	T	F	F	T	T	F	F	T	T	T
F	F	T	T	F	T	F	F	T	T	T
F	F	F	T	T	F	F	F	T	T	T

Since the last three columns of $p \to (q \vee r)$, $\quad (p \wedge \sim q) \to r$, $\quad (p \wedge \sim r) \to q$ have the same truth values for all possible cases in the truth table, then these statement forms are logically equivalent.

(b) $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$

p	q	r	$p \wedge q$	$q \vee r$	$(p \wedge q) \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	T	F
F	T	F	F	T	F	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

$(p \wedge q) \vee r$ and $p \wedge (q \vee r)$ (the last two columns in the truth table above) have different truth values in the fifth and the seventh rows. So they are not logically equivalent.

(c)

$p \to (q \to r)$, $\quad (p \wedge q) \to r$

p	q	r	$q \to r$	$p \wedge q$	$p \to (q \to r)$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	F	T	T
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

$p \to (q \to r)$ and $(p \wedge q) \to r$ (the last two columns in the truth table above) have the same truth values in all rows of the truth table. So they are logically equivalent.

2. Determine whether $\big(p \to (q \to r)\big) \leftrightarrow \big((p \wedge q) \to r\big)$ is a tautology.

   Answer: From question 1 c), we have $p \to (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent, so they have the same truth values in all rows of the truth table. The biconditional statement is true when it connects with the statement forms with the same truth values. Hence, $\big(p \to (q \to r)\big) \leftrightarrow \big((p \wedge q) \to r\big)$ is always true and it is a tautology.

   Alternatively, we can construct the truth table below.

p	q	r	$q \to r$	$p \wedge q$	$p \to (q \to r)$	$(p \wedge q) \to r$	$\big(p \to (q \to r)\big) \leftrightarrow \big((p \wedge q) \to r\big)$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	T
T	F	T	T	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	T	F	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

Since the truth values of $\big(p \to (q \to r)\big) \leftrightarrow \big((p \wedge q) \to r\big)$ are all true from the truth table, then it is a tautology.

3. Construct truth tables for the statement form $p \wedge \sim r \leftrightarrow q \vee r$.

   Answer: The order of the connective operators has to be used here:

$$p \wedge \sim r \leftrightarrow q \vee r \equiv (p \ \wedge (\sim r)) \leftrightarrow (q \vee r).$$

p	q	r	$\sim r$	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	F	T

4. Suppose that p and q are statements so that $p \to q$ is false. Find the truth values of each of the following: (a) $\sim p \to q$ (b) $p \vee q$ (c) $q \to p$

   Answer: Since $p \to q$ is false only when p is true and q is false, then

p	q	$\sim p$	$\sim p \to q$	$p \vee q$	$q \to p$
T	F	F	T	T	T

   . That is,

   (a) $\sim p \to q$ (b) $p \vee q$ (c) $q \to p$ are all true.

5. Consider the following statement:

   Catching the 8:05 bus is a sufficient condition for my being on time for work.

   (a) Write the above statement in **if-then** form.
   (b) Write the **contrapositive**, **inverse**, and **converse** of the above statement.

   Answer:

   (a) If I catch the 8:05 bus, then I am on time for work.
   (b) **Contrapositive**: If I am not on time for work, then I do not catch the 8:05 bus.
   Inverse: If I do not catch the 8:05 bus, then I am not on time for work.
   Converse: If I am on time for work, then I catch the 8:05 bus.

6. Use truth tables to determine whether the argument forms are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer.

(a)
$$p \to q$$
$$q \to p$$
$$\therefore\ p \vee q$$

(b)
$$p$$
$$p \to q$$
$$\sim q \vee r$$
$$\therefore\ r$$

(c) If x is divisible by 9, then x is divisible by 3

x is divisible by 3

$\therefore$ x is divisible by 9

<u>Answer:</u>

(a)

		premises		conclusion
p	q	$p \to q$	$q \to p$	$p \vee q$
T	T	T	T	T
T	F	F	T	
F	T	T	F	
F	F	T	T	F

This row shows that it is possible for an argument of this form to have true premises and a false conclusion. Thus this argument form is invalid. (← row F F)

(b)

				premises			conclusion
p	q	r	$\sim q$	p	$p \to q$	$\sim q \vee r$	r
T	T	T	F	T	T	T	T
T	T	F	F	T	T	F	
T	F	T	T	T	F	T	
T	F	F	T	T	F	T	
F	T	T	F	F	T	T	
F	T	F	F	F	T	F	
F	F	T	T	F	T	T	
F	F	F	T	F	T	T	

This row describes the only situation in which all the premises are true. Because the conclusion is also true here, the argument form is valid. (← row T T T)

(c)
Let p be the statement " x is divisible by 9" and q be the statement "x is divisible by 3. " Then

$$\begin{array}{ll} & p \to q \\ & q \\ \therefore & p \end{array}$$

Truth table:

p	q	$p \to q$	q	p	
T	T	T	T	T	←-- critical row
T	F	F	F		
F	T	T	T	F	←-- critical row: This has true premises & false conclusion.
F	F	T	F		

Notice that the columns 3 and 4 consist of the truth values of premises. Since the third row (which is a critical row) has true premises with false conclusion, then this argument is **invalid**.

7. Suppose that you discover a note written by a pirate. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (a-e below) and challenged the reader to use them to figure out the location of the treasure.

   a. If this house is next to a lake, then the treasure is not in the kitchen.
   b. If the tree in the front yard is a lemon tree, then the treasure is in the kitchen.
   c. This house is next to a lake.
   d. The tree in the front yard is a lemon tree or the treasure is buried under the flagpole.
   e. If the tree in the back yard is a coconut tree, then the treasure is in the garage.

   Where is the treasure hidden?

   Answer: Define the following statement variables.
   HL = This house is next to a lake.
   TK = The treasure is in the kitchen.
   YL = The tree in the front yard is a lemon tree.
   TP = The treasure is buried under the flagpole.
   YC = The tree in the back yard is a coconut tree.
   TG = The treasure is in the garage.

   (1) HL $\rightarrow\sim$ TK by (a)
   HL by (c)
   $\therefore \sim$ TK
   (2) YL $\rightarrow$ TK by (b)
   $\sim$ TK by by the conclusion of (1)
   $\therefore \sim$ YL by "modus tollens"
   (2) YL $\vee$ TP by (b)
   $\sim$ YL by the conclusion of (2)
   $\therefore$ TP by "elimination"

   Therefore, we conclude that the treasure is buried under the flagpole.