

Exercise IV

1. Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{v_1, v_2\}$ is a basis for $\mathbb{R}^3$. Is $\{v_1, v_2\}$ a basis for $\mathbb{R}^2$?

2. Let $v_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, $v_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$.

3. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Then every vector in H is a linear combination of v_1 and v_2 because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Is $\{v_1, v_2\}$ a basis for H ?

4. Let $v_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, and $H = \text{Span}\{v_1, v_2, v_3\}$. It may be verified that $4v_1 + 5v_2 - 3v_3 = 0$. Use this information to find a basis for H . There is more than one answer.

5. Let $v_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$. It may be verified that $v_1 - 3v_2 + 5v_3 = 0$. Use this information to find a basis for $H = \text{Span}\{v_1, v_2, v_3\}$.

6. Suppose $\mathbb{R}^4 = \text{Span}\{v_1, \dots, v_4\}$. Explain why $\{v_1, \dots, v_4\}$ is a basis for $\mathbb{R}^4$.

Decide whether each statement is true or false, and give a reason for each answer. Here V is a nonzero finite-dimensional vector space.

7. If $\dim V = p$ and if S is a linearly dependent subset of V , then S contains more than p vectors.
8. If S spans V and if T is a subset of V that contains more vectors than S , then T is linearly dependent.

Solution Part 1

1. Let $A = [v_1 \ v_2]$. Row operations show that

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

Not every row of A contains a pivot position. So the columns of A do not span $\mathbb{R}^3$, by Theorem 2 in Section 2.2. Hence $\{v_1, v_2\}$ is not a basis for $\mathbb{R}^3$. Since v_1 and v_2 are not in $\mathbb{R}^2$, they cannot possibly be a basis for $\mathbb{R}^2$. However, since v_1 and v_2 are obviously linearly independent, they are a basis for a subspace of $\mathbb{R}^3$, namely, $\text{Span}\{v_1, v_2\}$.

2. Set up a matrix A whose column space is the space spanned by $\{v_1, v_2, v_3, v_4\}$, and then row reduce A to find its pivot columns.

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two columns of A are the pivot columns and hence form a basis of $\text{Col } A = W$. Hence $\{v_1, v_2\}$ is a basis for W . Note that the reduced echelon form of A is not needed in order to locate the pivot columns.

3. Neither v_1 nor v_2 is in H , so $\{v_1, v_2\}$ cannot be a basis for H . In fact, $\{v_1, v_2\}$ is a basis for the *plane* of all vectors of the form $(c_1, c_2, 0)$, but H is only a *line*.
4. The three simplest answers are $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_2, v_3\}$. Other answers are possible.
5. the set $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_1, v_3\}$ are linearly independent and thus each form a basis for H .

6. **Hint: Use the Invertible Matrix Theorem.**

Let $A = [v_1 \ v_2 \ v_3 \ v_4]$ then A is a square and its columns span $\mathbb{R}^4$ since $\mathbb{R}^4 = \text{span}\{v_1 \ v_2 \ v_3 \ v_4\}$. So its columns are linearly independent by the invertible matrix theorem and $\{v_1 \ v_2 \ v_3 \ v_4\}$ is a basis for $\mathbb{R}^4$.

7. False. Consider the set $\{0\}$.
8. True. By the Spanning Set Theorem, S contains a basis for V ; call that basis S' . Then T will contain more vectors than S' . By Theorem 10, T is linearly dependent.

Part 2

The matrices below are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find $\text{rank } A$ and $\dim \text{Nul } A$.
2. Find bases for $\text{Col } A$ and $\text{Row } A$.
3. What is the next step to perform if one wants to find a basis for $\text{Nul } A$?
4. How many pivot columns are in a row echelon form of A^T ?

Part 2 solutions

1. A has two pivot columns, so $\text{rank } A = 2$. Since A has 5 columns altogether, $\dim \text{Nul } A = 5 - 2 = 3$.
2. The pivot columns of A are the first two columns. So a basis for $\text{Col } A$ is

$$\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$$

The nonzero rows of B form a basis for $\text{Row } A$, namely, $\{(1, -2, -4, 3, -2), (0, 3, 9, -12, 12)\}$. In this particular example, it happens that any two rows of A form a basis for the row space, because the row space is 2-dimensional and none of the rows of A is a multiple of another row. In general, the nonzero rows of an echelon form of A should be used as a basis for $\text{Row } A$, not the rows of A itself.

3. For $\text{Nul } A$, the next step is to perform row operations on B to obtain the reduced echelon form of A .
4. $\text{Rank } A^T = \text{rank } A$, by the Rank Theorem, because $\text{Col } A^T = \text{Row } A$. So A^T has two pivot positions.

CHAPTER 5 SUPPLEMENTARY EXERCISES

1. Mark each statement as True or False. Justify each answer. In parts (a)–(f), $v_1, \dots, v_p$ are vectors in a nonzero finite-dimensional vector space V , and $S = \{v_1, \dots, v_p\}$.

- ___ a. The set of all linear combinations of $v_1, \dots, v_p$ is a vector space.
- ___ b. If $\{v_1, \dots, v_{p-1}\}$ spans V , then S spans V .
- ___ c. If $\{v_1, \dots, v_{p-1}\}$ is linearly independent, then so is S .
- ___ d. If S is linearly independent, then S is a basis for V .
- ___ e. If $\text{Span } S = V$, then some subset of S is a basis for V .
- ___ f. If $\dim V = p$ and $\text{Span } S = V$, then S cannot be linearly dependent.
- ___ g. A plane in $\mathbb{R}^3$ is a two-dimensional subspace.
- ___ h. The nonpivot columns of a matrix are always linearly dependent.
- ___ i. Row operations on a matrix A can change the linear dependence relations among the rows of A .
- ___ j. Row operations on a matrix can change the null space.
- ___ k. The rank of a matrix equals the number of nonzero rows.
- ___ l. If an $m \times n$ matrix A is row equivalent to an echelon matrix U , and if U has k nonzero rows, then the dimension of the solution space of $Ax = 0$ is $m - k$.
- ___ m. If B is obtained from a matrix A by several elementary row operations, then $\text{rank } B = \text{rank } A$.
- ___ n. If A is $m \times n$ and $\text{rank } A = m$, then the linear transformation $x \mapsto Ax$ is one-to-one.
- ___ o. If A is $m \times n$ and the linear transformation $x \mapsto Ax$ is onto, then $\text{rank } A = m$.
- ___ p. A change-of-coordinates matrix is always invertible.
- ___ q. If $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ are bases for a vector space V , then the j th column of the change-of-coordinates matrix $c_{\mathcal{C}}^{\mathcal{B}}$ is the coordinate vector $[c_j]_B$.

2. Find a basis for the set of all vectors of the form

$$\begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{bmatrix} \quad (\text{Be careful.})$$

3. Let $u_1 = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, and $W = \text{Span}\{u_1, u_2\}$. Find an *implicit* description of W : that is, find a set of one or more homogeneous equations that characterize the points of W . [Hint: When is b in W ?]
4. Suppose p_1, p_2, p_3, p_4 are specific polynomials that span a two-dimensional subspace H of $\mathcal{P}_5$. Describe how one can find a basis for H by examining the four polynomials and making almost no computations.
5. What would you have to know about the solution set of a homogeneous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss.
6. Let H be an n -dimensional subspace of an n -dimensional vector space V . Explain why $H = V$.
7. Let S be a maximal linearly independent subset of a vector space V . That is, S has the property that if a vector not in S is adjoined to S , then the new set will no longer be linearly independent. Prove that S must be a basis for V . [Hint: What if S were linearly independent but not a basis of V ?]
8. Let S be a finite minimal spanning set of a vector space V . That is, S has the property that if a vector is removed from S , then the new set will no longer span V . Prove that S must be a basis for V .

Exercises 9–12 develop properties of rank that are sometimes needed in applications. Assume the matrix A is $m \times n$.

9. Show that if P is an invertible $m \times m$ matrix, then $\text{rank } PA = \text{rank } A$. [Hint: Explain why P is a product of elementary matrices, and show that PA is row equivalent to A .]