EE320 (2/2012)

INTRODUCTORY MATHEMATICAL ECONOMICS

MATRIX ALGEBRA AND ITS APPLICATION

(Part 1 – Basic Matrix Algebra)

Topics

- Why do economists need matrix algebra?
- Representation of a system of equations by matrix notations
- Review matrices and matrix operation
  - Types of matrix
  - Matrix operations
- Determinant and singularity of matrix
- Matrix inversion by determinant
- Cramer's rule

Why do economists need matrix algebra?

- Suppose you are asked to solve the following system of simultaneous equations:

$$Q_{d1} = 8 - 3P_1 + P_2 - 4P_3, \quad Q_{s1} = 2 + 2P_1, \quad Q_{d1} = Q_{s1}$$
$$Q_{d2} = 12 + 5P_1 - 2P_2 + 2P_3, \quad Q_{s2} = 2 + 3P_2, \quad Q_{d2} = Q_{s2}$$
$$Q_{d3} = 7 - 6P_1 + P_2 - 3P_3, \quad Q_{s3} = 5 + 4P_3, \quad Q_{d3} = Q_{s3}$$

Question: How would you derive the equilibrium prices and quantities for the 3 goods?

- Linear algebra: solution by elimination of variables

  ➔ This could be quite tedious and prone to mistakes.

- Matrix algebra?

Why do economists need matrix algebra? (Cont'd)

- Matrix algebra can be very useful in handling a large system of simultaneous equations, commonly used in economics.
- Why?
  - It provides a compact way of writing an equation system.
  - It leads to a way of testing the existence of a solution by evaluation of a *determinant*.
  - It gives a method of finding the solution (if exists).
- Drawback: Matrix algebra is applicable only to linear-equation system.

System of Equations in Matrix Form

- Given a system of m linear equations in n variables:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = d_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = d_2$$
$$\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = d_m$$

- We can form a matrix from the above system of equations as: $\boldsymbol{AX = d}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}] \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_m \end{bmatrix}$$

Coefficients Variables Constant terms

System of Equations in Matrix Form (Cont'd)

- Example:

$$Q_d = Q_s$$
$$Q_d = a - bP$$
$$Q_s = -c + dP$$

→Write in matrix form AX = d:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ -a \\ c \end{bmatrix}$$

- Later on we will show that, if $|A| \neq 0$, then $X = A^{-1}d$.

Example

- Suppose a firm sells 3 products in 3 regions.

Region	**Sale Volume**		
	Good A	Good B	Good C
North	100	80	90
Central	50	110	70
South	120	60	130

- Matrix of sale volumes can be written as:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix}$$

Types of Matrices

- Let A be an $m \times n$ matrix, where m is the number of rows and n is the number of columns.

1. When $m=n$, matrix A is a square matrix.
2. When $n=1$, matrix A is a column vector.
3. When $m=1$, matrix A is a row vector.
4. An identity matrix is a square matrix containing ones along the diagonal and zeros elsewhere:

$$I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

5. A null matrix is a matrix whose elements are all zero.
6. A diagonal matrix is a square matrix $A_{nxn} = [a_{ij}]$ with $a_{ij}=0$ when $i \neq j$.

Basic Matrix Operations

- Let A and B be m x n matrices.

1. $A = B$ if and only if $a_{ij} = b_{ij}$ for all values of i and j.
2. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then

$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$$

$$A - B = [a_{ij}]_{m \times n} - [b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$$

Example:

$$\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} =$$

3. If α is a real number, then $\alpha A = \alpha[a_{ij}]_{m \times n} = [\alpha a_{ij}]_{m \times n}$

Example:

$$5\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} =$$

Basic Matrix Operations (cont'd)

4. Rules of Matrix addition and multiplication by scalars
   a) $(A + B) + C = A + (B + C)$
   b) $A + B = B + A$
   c) $A + 0 = A$
   d) $A + (-A) = 0$
   e) $(\alpha + \beta)A = \alpha A + \beta A$
   f) $\alpha (A + B) = \alpha A + \alpha B$

Matrix Multiplication

- Suppose $A = [a_{ij}]_{m\ x\ n}$ and $B = [b_{ij}]_{n\ x\ p}$.

Then, the product $C = AB$ is the $n\ x\ p$ matrix $\mathbf{C = [c_{ij}]_{n\ x\ p}}$, whose element ij is the inner product

$$C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

- Example:

Let $A_{1x2} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ $B_{2x3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

➔ $C = AB =$

Example: Matrix Multiplication

- Suppose the sale volume matrix and the price vector are given by:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \qquad P = \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

- The total revenue from the sales of all products is:

$$TR = QP = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} =$$

Rule for Matrix Multiplication:

a) $AB \neq BA$ *(in general)*

b) $(AB)C = A(BC)$

c) $(A + B)C = AB + BC$

d) $(B + C)A = BA + CA$

- Example:

Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$ $B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$

→ $AB =$

→ $BA =$

Matrix Transposition

- The transpose of any *m x n* matrix *A*, denoted by *A′* or A^T, is defined as the *n x m* matrix whose first column is the first row of *A*, and the second column is the second row of *A*, and so on.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \longrightarrow A' = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

- Example:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \longrightarrow Q' = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Properties of transposes

a) $(A')' = A$

b) $(A + B)' = A' + B'$

c) $(AB)' = B'A'$

d) $(\alpha A)' = \alpha A'$

Note: A square matrix A is symmetric (i.e. $A = A'$) iff $a_{ij} = a_{ji}$ for all i,j

- Example:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$$

→ $(AB)' =$

→ $B'A' =$

Matrix Inversion

- The inverse of matrix A, denoted by A^{-1}, is defined only if A is a square matrix and must satisfy the condition

$$AA^{-1} = A^{-1}A = I.$$

- Properties of the inverse:
  a) If A^{-1} exists, then $(A^{-1})^{-1} = A$.
  b) If AB is invertible, then $(AB)^{-1} = B^{-1}A^{-1}$
  c) $(A')^{-1} = (A^{-1})'$
  d) $(cA)^{-1} = c^{-1}A^{-1}$, where $c \neq 0$

- If a square matrix A has an inverse, A is said to be nonsingular. If A is not invertible, then A is called a singular matrix.
  - The nonsingularity condition is required for a solution of a linear-equation system to exist.

Inverse Matrix and Solution of Linear-Equation System

- Given the equation system in matrix notation:

$$AX = d$$

$$A^{-1}AX = A^{-1}d$$

$$\Rightarrow \quad X = A^{-1}d$$

where X is the column vector of variables, and $A^{-1}d$ is the column vector of solution values.

  - Since A^{-1}, if it exists, is unique, the solution $A^{-1}d$ must be unique values.

- Example:

$$\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -a \\ c \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -a \\ c \end{bmatrix}$$

- Next, we will study how to determine A^{-1} by determinant.

Determinant

- One way to test whether an inverse of a matrix exists (i.e. it is nonsingular) is to use determinant.
- The determinant of a *square* matrix A, denoted by |A|, is a uniquely defined associated with that matrix.
  - Determinant of order 1: |A| = |a| = a
  - Determinant of order 2: $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
  - Determinant of order 3:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: Determinants of Order 3

- Evaluate the following determinant:

$$|A| = \begin{vmatrix} 5 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} =$$

Evaluating an *n*th-Order Determinants by Laplace Expansion

<u>Definitions</u>: Let A be an n x n matrix.

- The minor of the element a_{ij} is: $|M_{ij}| = \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$

- The cofactor of the element a_{ij} is: $|C_{ij}| = (-1)^{i+j}\,|M_{ij}|$

- The determinant of the matrix A of order n can be found by the Laplace expansion of *any row* or *any column* as follows:

$$|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion by the } i\text{th row]}$$

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion by the } j\text{th row]}$$

Basic Properties of Determinants (1)

- Property I

The interchange of rows and columns does not affect the value of a determinant. That is, |A'| = |A|.

Example:

$$|A| = \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} = \qquad |A'| = \begin{vmatrix} 3 & 4 \\ 1 & 7 \end{vmatrix} =$$

- Property II

The interchange of any two rows (or columns) will alter the sign, but not the numerical value of the determinant.

Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \qquad |B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} =$$

Basic Properties of Determinants (2)

- <u>Property III</u>

If all element in any one row (or column) is multiplied by a scalar k, the determinant is multiplied by k.

Example: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} =$ $\qquad |B| = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} =$

- <u>Property IV</u>

The addition (or subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered.

Example: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} =$ $\qquad |B| = \begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} =$

Basic Properties of Determinants (3)

- Property V

If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example:

$$[A] = \begin{bmatrix} 3 & 4 & 2 \\ 15 & 20 & 10 \\ 4 & 0 & 1 \end{bmatrix}$$

→ |A| =

Note:

The condition of *linear independence* is a sufficient condition for the *nonsingularity* of a matrix. For the rows (or columns) to be linearly independent, none must be a linear combination of the rest.

Determinantal Criterion for Nonsingularity

- Summary:

$|A| \neq 0$ $\Leftrightarrow$ there is row (column) independence in matrix A

A is nonsingular

A^{-1} exists

a unique solution $X^* = A^{-1}d$ exists

Matrix Inversion by Determinant

- Assume that A is an n x n nonsingular matrix, and A|≠ 0.
- The inverse of matrix A is:

$$A^{-1} = \frac{1}{|A|} adj(A)$$

where

$$adj(A) \equiv C'_{nxn} \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}$$

and

$$|C_{ij}| = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

Steps to Find the Inverse of a Matrix A

1. Find the determinant |A|. If |A| = 0, then the inverse does not exist.
2. Find the cofactors of all the elements of A, and arrange them as a matrix: $C = [|C_{ij}|]$
3. Take the transpose of the cofactor matrix, *C*, to get the *adj(A)*.
4. Divide the adj(A) by the determinant.

Example:

Find the inverse of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$$

Cramer's Rule

- Given an equation system $Ax = d$, where A is an n x n nonsingular matrix, the solution value of the jth variable can be obtained from:

$$x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|}\begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix}$$

where $|A_j|$ is the determinant of the matrix A when the jth column is replaced by the constant terms $d_1 \ldots d_n$.

Example: Cramer's Rule

- Given the system of two equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

- By Cramer's rule, the solutions are given by: