8 Introduction to Calculus of Several Variables

8.1 Function of Two or More Variables

Typical relationships in economics are relationships of more than one variable, for example,

$$z = f(x, y)$$

Here, z is a dependent variable which is a function of two variables x and y where x and y are independent variables. Hence, the values of z is dependent on the values of x and y. The domain of z function is not stated then the domain can be said to be a set of all x and y for which the expression $f(x, y)$ is defined (*i.e.* is real number).

Ex. 1:

(a) $f(x, y) = 2x + x^2 y^3$ Find the domain of f, $f(1,0)$, $f(0,1)$ and $f(a+1, b)$.

[Ans: $x = \Re, y = \Re$, 2, 0, $2(a+1) + (a+1)^2 b^3$]

(b) $f(x, y) = \dfrac{3x^2 + 5y}{x - y}$ Find the domain of f and $f(1,-2)$.

[Ans: $x \neq y$, -7/3]

(c) $f(x, y) = xe^y + \ln x$ Find the domain of f and $f(e^2, \ln 2)$.

[Ans: $x > 0$, $2\left(e^2 + 1\right) \approx 16.78$

(d) $f(x, y, z) = xy + xz + yz$ Find $f(-1,2,5)$.

[Ans: 3]

Ex. 2: Production function $F(x, y) = Ax^a y^b$ (Function in this form is called ***'Cobb-Douglas'*** function) where A, a and b are constants. F is number of units produced whereas x and y are input factors.

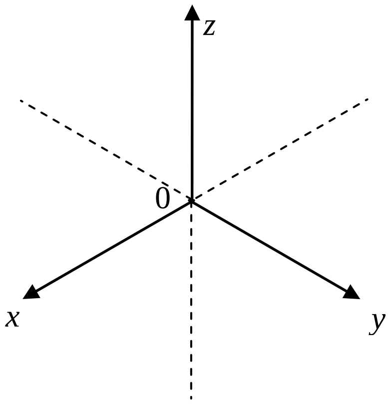
Show that

$$F(2x, 2y) = 2^{a+b} F(x, y)$$

And

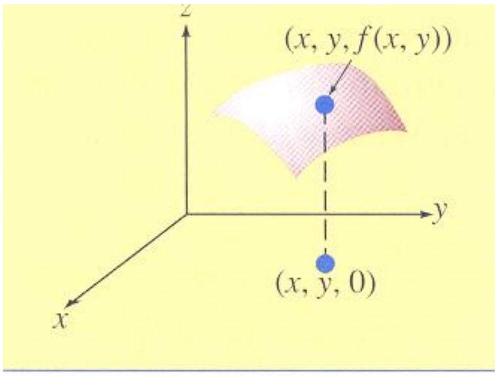
$$F(x+h, y) - F(x, y) = Ay^b \left\{(x+h)^a - x^a\right\}.$$

8.2 Geometrical Representation of Function of Two Variables



The graph of a two-variable function is the set of triple (x, y, z) where (x, y) is in the domain of f and $z = f(x, y)$. $z = f(x, y)$ can be plotted in the three-dimensional rectangular coordinate system.

- The origin of the system is where the *x*-, *y*- and *z*-axes intercepts and all three axes are perpendicular.
- The arrows indicate the positive direction.
- The negative portion of the axes shows as dashed lines.
- Function of two variables can be represented by a 3-D plot.
- Function of more than two variables cannot be visualised.
- The plot of a two-variable function becomes a surface in three-dimensional space.



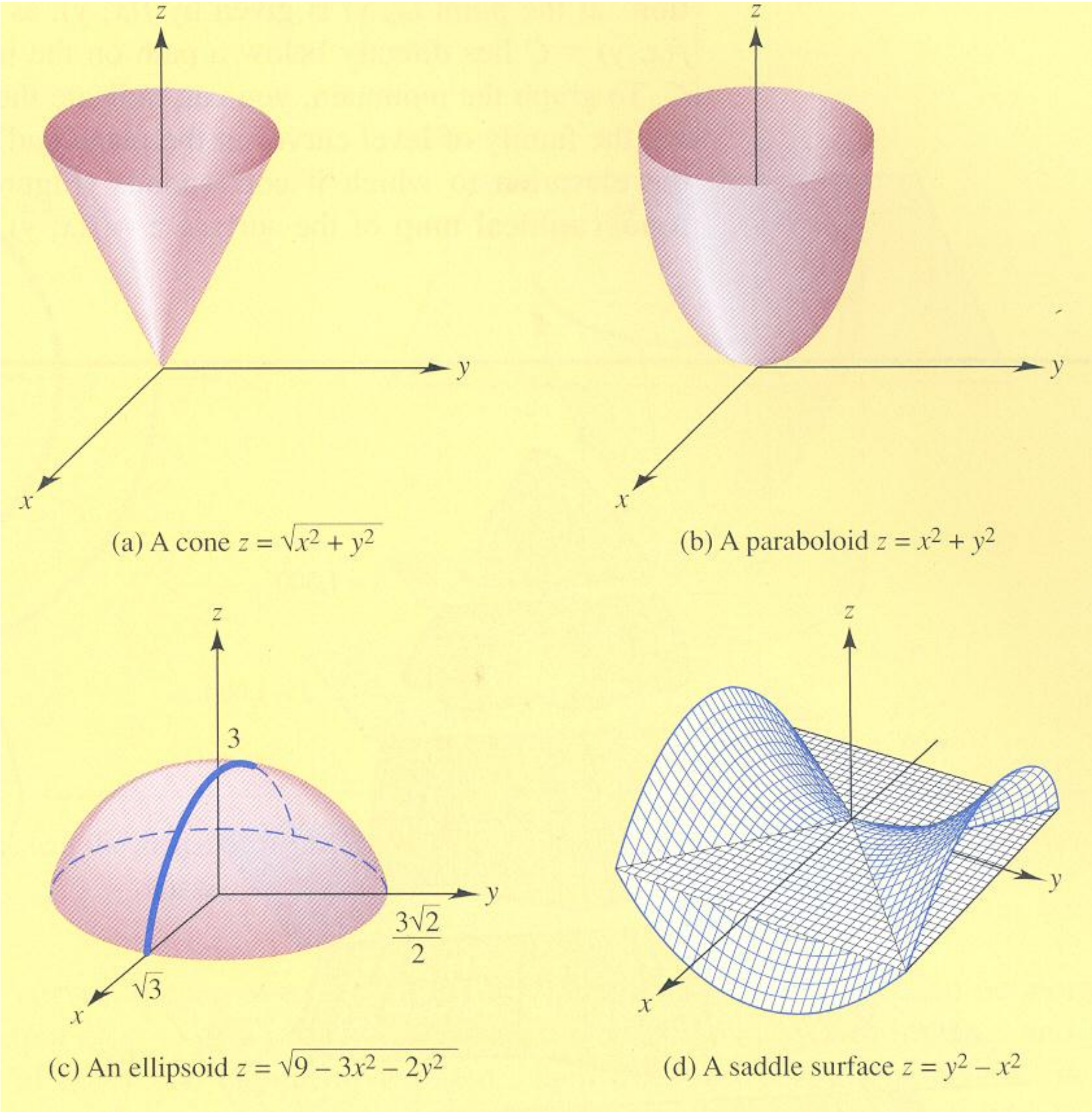


Figure 8.1 Example of surfaces represented by two-variable functions

8.2.1 Level Curves

The **level curve** *f* at a *C* (*C* is a constant) is defined as the set of points with coordinates (x, y) in the plane *xy* that satisfy $f(x, y) = C$. An entire family of level curves is generated as *C* varies over a set of numbers. These level curves can help to visualise the surface plot.

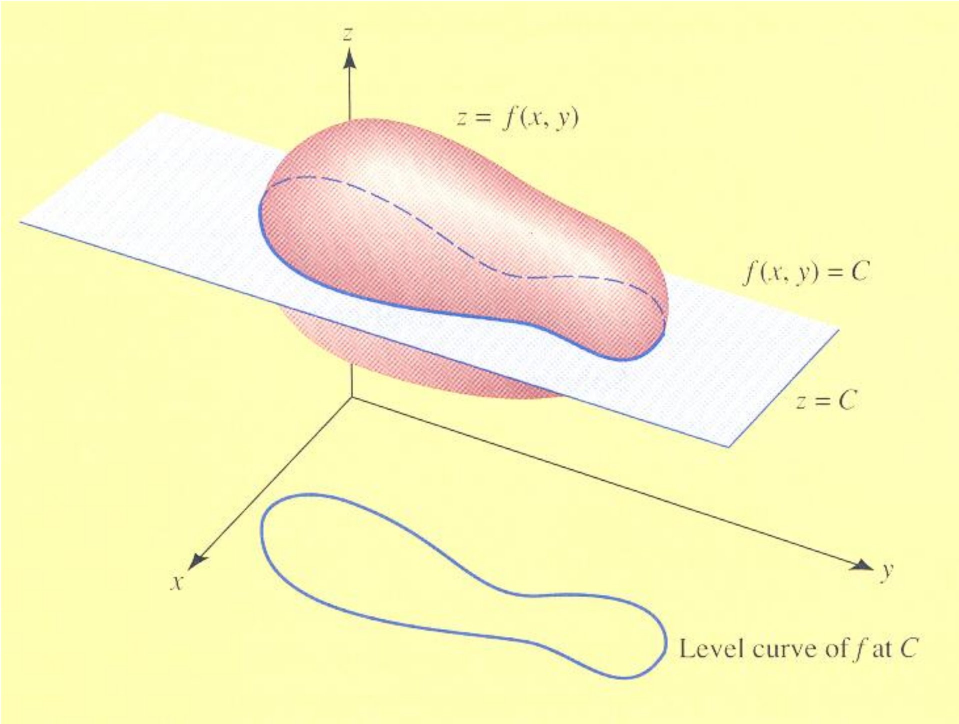


Figure 8.2 Level curve of *f* at *C*

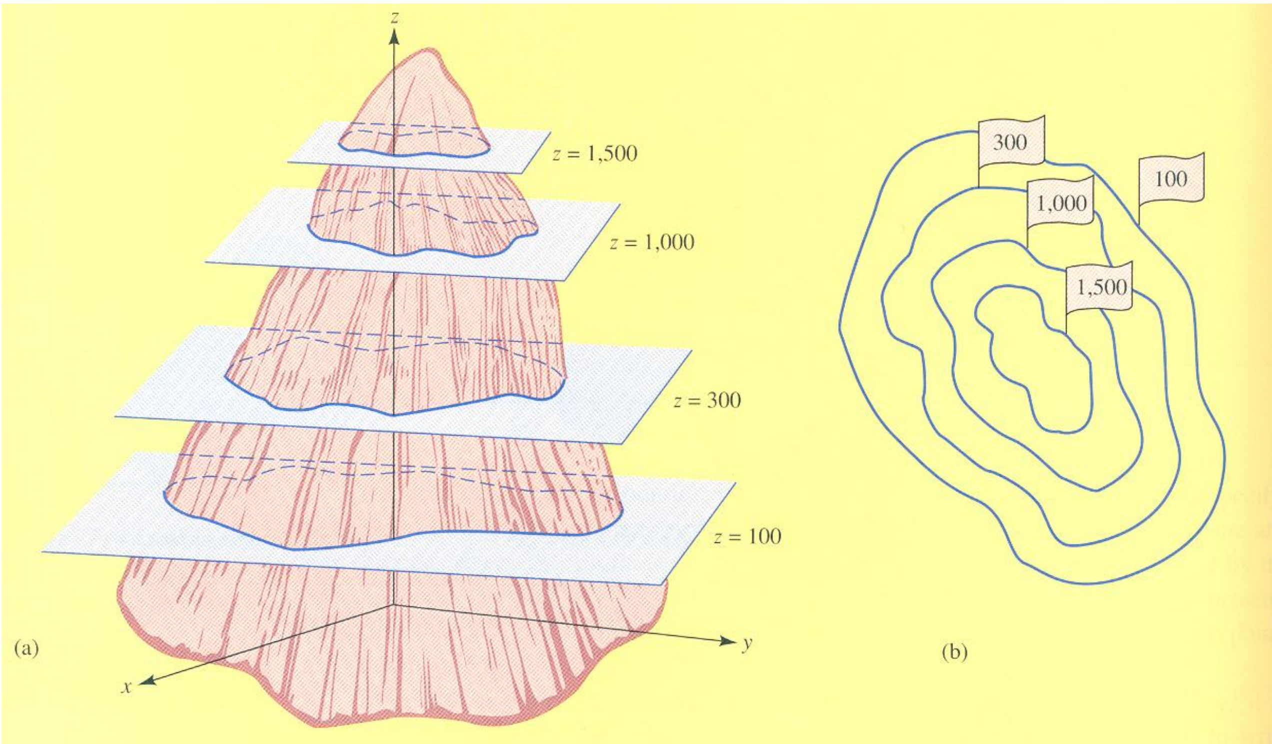
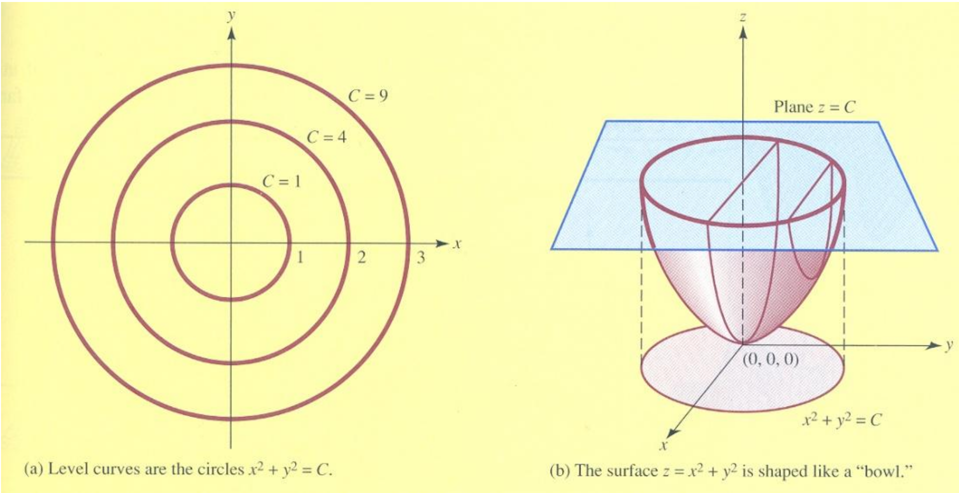
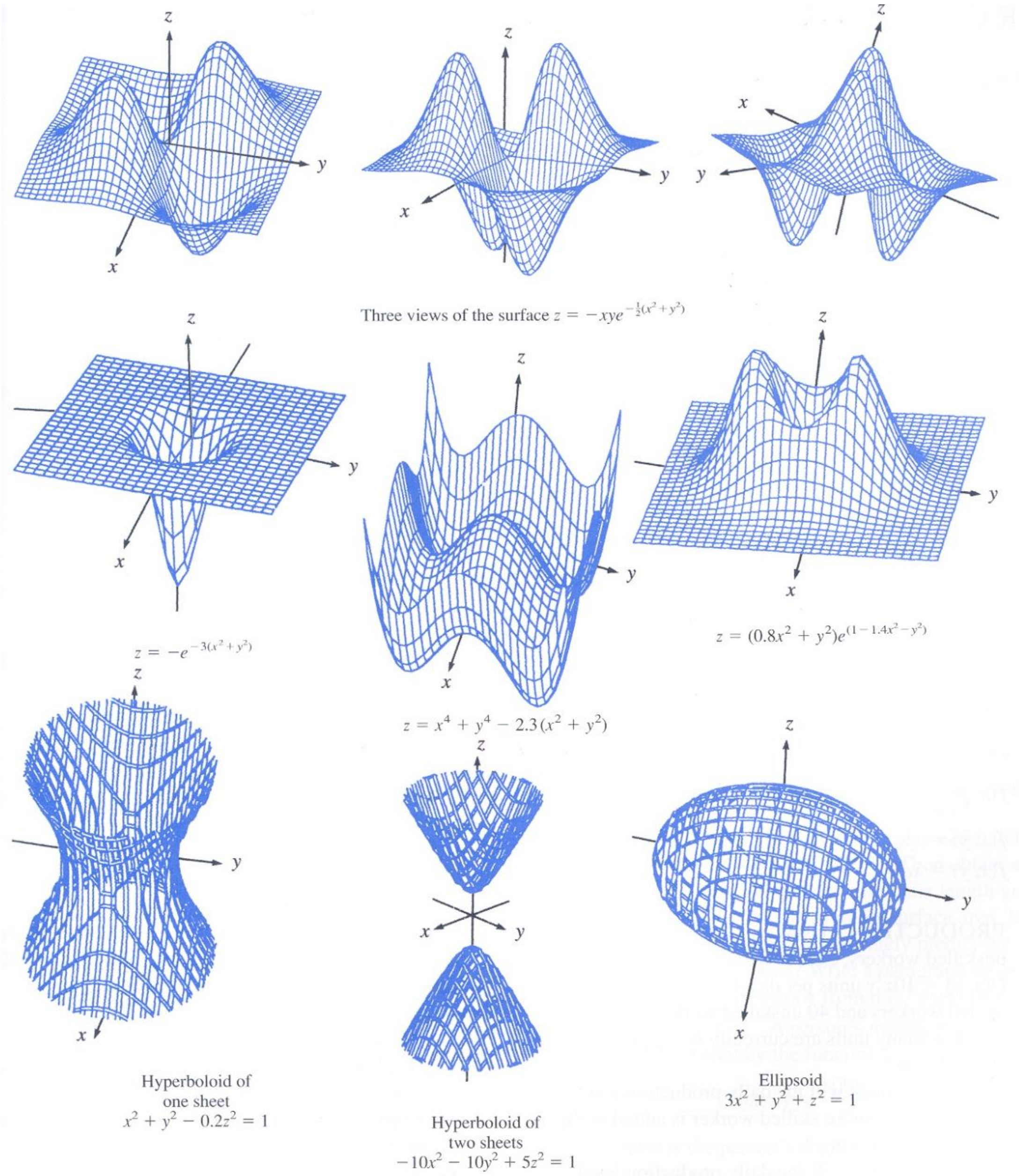


Figure 8.3 Application of level curves in creating a topographical map for mountain



(a) Level curves are the circles $x^2 + y^2 = C$. (b) The surface $z = x^2 + y^2$ is shaped like a "bowl."

Figure 8.4 A family of level curves in (a) help to visualise the shape of a surface (b).



8.3 Partial Derivatives

Recall for a single variable function $y = f(x)$, $\dfrac{dy}{dx}$ is a rate of change of y when x changes.

The process of obtaining $\dfrac{dy}{dx}$ is called "differentiation". Now having a two-variable function, $z = f(x, y)$, if we want to know how z changes when x **or** y changes, we use partial derivatives.

For example, for a function: $z = x^3 + 2y^2$

The rate of change of z **with respect to** x only can be found by differentiating z with respect to x while treating y as a constant. The rate of change of z with respect to x only is called the partial derivative of z with respect to x. Hence,

$$\frac{\partial z}{\partial x} = 3x^2$$

Note: We use the notation "∂" not "d" to indicate partial derivative.

Similarly, **the rate of change of** z **with respect to** y can be found by differentiating z with respect to y while treating x as a constant. Hence,

$$\frac{\partial z}{\partial y} = 4y$$

8.3.1 Partial Derivative Notations

- Partial derivative of f or z with respect to x:

$$f_x'(x, y),\ f_x(x, y), \frac{\partial}{\partial x}\left[f(x, y)\right] \text{ and } \frac{\partial z}{\partial x}$$

- Partial derivative of f or z with respect to x evaluated at (x_0, y_0):

$$f_x'(x_0, y_0),\ f_x(x_0, y_0), \left[\frac{\partial z}{\partial x}\right]_{(x_0, y_0)}, \left[\frac{\partial z}{\partial x}\right]_{\substack{x=x_0\\ y=y_0}}, \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} \text{ and } \left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}}$$

Ex. 3: $f(x, y) = x^3 y + x^2 y^2 + x + y^2$ Find all partial derivatives of $f(x, y)$.

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial f}{\partial y} =$$

Ex. 4:

(a) $f(x, y) = \frac{xy}{x^2 + y^2}$ Find all partial derivatives of $f(x, y)$.

[Ans: $\frac{\partial f}{\partial x} = \frac{y^3 - x^2 y}{\left(x^2 + y^2\right)^2}$ and $\frac{\partial f}{\partial y} = \frac{x^3 - xy^2}{\left(x^2 + y^2\right)^2}$]

(b) $z = \left(x^2 + xy + y\right)^5$ Find all partial derivatives.

[Ans: $\frac{\partial z}{\partial x} = 5\left(x^2 + xy + y^4\right)\left(2x + y\right)$ and $\frac{\partial z}{\partial y} = 5\left(x^2 + xy + y^4\right)\left(x + 1\right)$]

Ex. 5: A demand for rice is $x = \frac{Am^{2.08}}{p^{1.5}}$ where x is the rice consumption, m is the income per family, p is the price, and A is a constant. Calculate $\frac{\partial x}{\partial p}$ and $\frac{\partial x}{\partial m}$.

8.4 Formal Definition of Partial Derivatives

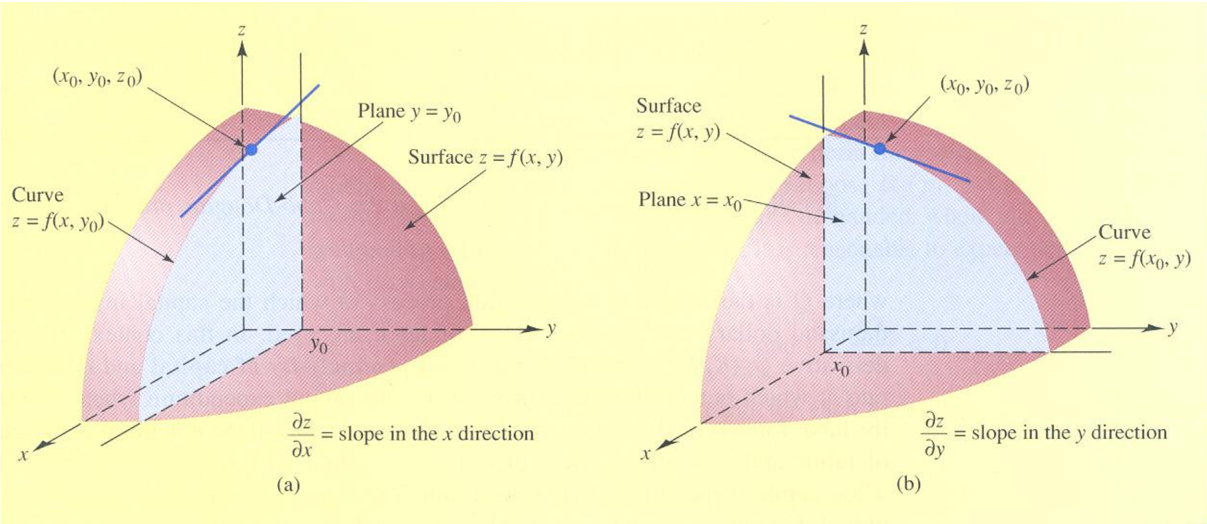


Figure 8.5 Graphical representations of (a) $\frac{\partial z}{\partial x}$ and (b) $\frac{\partial z}{\partial y}$.

According to Figure 8.5(a), slope of tangent line at $(x_0, y_0, f(x_0, y_0))$ = partial derivative of z (or f) with respect to x at (x_0, y_0)

By definition, $$\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} = f_x'(x_0,y_0) = \lim_{h\to 0}\left(\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}\right)$$

Similarly, $$\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} = f_y'(x_0,y_0) = \lim_{h\to 0}\left(\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}\right)$$

Ex. 6: A company produces 2 products, the joint cost function per week is given by $C = f(x,y) = 0.07x^2 + 75x + 85y + 6000$. Determine marginal cost $\frac{\partial C}{\partial x}$ and $\frac{\partial C}{\partial y}$ when $x = 100$ and $y = 50$. [Ans: 89,85]

Solution: $\frac{\partial C}{\partial x} =$ $\qquad$ $\frac{\partial C}{\partial y} =$

$\left[\frac{\partial C}{\partial x}\right]_{(100,50)} =$ $\qquad$ $\left[\frac{\partial C}{\partial y}\right]_{(100,50)} =$

8.5 Implicit Partial Differentiation

Sometimes, z is not given in term of x and y, for example,

$$z^2 - x^2 - y^2 = 0$$

Hence, in order to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, implicit partial differentiation is used as follows:

- Partial differentiate both sides of the equation with respect to x to find $\frac{\partial z}{\partial x}$.

$$2z\frac{\partial z}{\partial x} - 2x + 0 = 0$$

Rearranging, $$\frac{\partial z}{\partial x} = \frac{x}{z}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y}{z}$$

Ex. 7: If $\frac{xz^2}{x+y} + y^2 = 0$, evaluate $\frac{\partial z}{\partial x}$ when x = -1, y = 2, and z = 2. [Ans: 2]

Ex. 8: If $z = f(x, y)$ and $e^{yz} = -xyz$, find $\frac{\partial z}{\partial x}$ where $x = -\frac{e^2}{2}$, $y = 1$ and $z = 2$. [Ans: $-\frac{4}{e^2}$]

Ex. 9: If $se^{r^2+u^2} = u\ln(t^2+1)$ where s is a constant but r, u and t are variables, show that

$$\frac{\partial t}{\partial u} = \frac{(t^2+1)\left\{2sue^{r^2+u^2} - \ln(t^2+1)\right\}}{2ut}$$

8.6 Higher-Order Partial Derivative

For $z = f(x,y)$,

1st order partial derivatives are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2nd order partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

Note: Beyond the 2nd order derivatives can also be determined *e.g.*

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right] = \frac{\partial^3 f}{\partial y \partial x^2} = f_{xxy}$$

Ex. 10: Find the four second-order partial derivatives of $f(x,y) = 7x^2 + 3y$.

[Ans: $\frac{\partial^2 f}{\partial x^2} = 14$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial x \partial y} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = 0$]

Ex. 11: For $f(x,y) = x^2 y + x^2 y^2$, find f_y, f_{yy} and f_{yyx}.

[Ans: $f_{xx} = 2y + 2y^2$, $f_{xy} = 2x + 4xy$, $f_{yy} = x^2 + 2x^2 y$, $f_{yx} = 2x + 4xy$]

Ex. 12: For $f(x,y) = (x+y)^2 (xy)$, find f_x, f_y, f_{xx}, f_{yy}, f_{yyx} and f_{xyy}.

[Ans: $3x^2 y + 4xy^2 + y^3,\ x^3 + 4x^2 y + 3xy^2,\ 6xy + 4y^2, 4x^2 + 6xy,\ 8x + 6y,\ 8x + 6y$]

Ex. 13: For $f(x,y) = y^2 e^x + \ln(xy)$, find $f_{xyy}(1,1)$. [Ans: 2*e*]

8.7 Young's Theorem

All the m^{th}-order partial derivatives of the function $f(x_1, x_2, ..., x_n)$ are continuous. If any two involve differentiating with respect to each of the same variables the same number of times, then they are necessary equal. For example, if $m = 2$ and $z = f(x, y)$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}} \qquad \text{Provided that they are continuous.}$$

Ex. 14: For $z = f(x, y) = 2x^4 + 3x^3y^3 + xy^2 + y$**,** show that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = 27x^2y^2 + 2y$.

Ex. 15: C is the total cost of producing q_C units of digital cameras and q_F units of memory sticks is: $C = 30q_C + 0.015q_Cq_F + q_F + 900$.

The demand function is $q_C = \dfrac{9000}{p_C\sqrt{p_F}}$ and $q_F = 2000 - p_C - 400p_F$ where p_C is the price per digital camera and p_F is the price per memory sticks. Find the rate of change of the total cost with respect to price of digital camera when the price per digital camera is 50 and the price per memory stick is 2. [Ans: -123.2]

8.8 Chain Rule

$$z = f_1(x, y),\ x = f_2(r, s) \text{ and } y = f_3(r, s)$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} \qquad \text{and} \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Note: In the chain rule, the number of intermediate variables of z (in this case is 2) is the same as the number of terms that compose each of $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial s}$.

Ex. 16: For $z = 5x + 3y, x = 2r + 3s$ and $y = r - 2s$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial s}$. [Ans: 13, 9]

8.8.1 Chain Rule (Extension)

Similar to the chain rule for single-variable functions,

- If $z = f(x,y)$, $x = g_1(r,s,t)$ and $y = g_2(r,s,t)$ then

  The partial derivative of z with respect to r is $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$

  The partial derivative of z with respect to s is $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$

  The partial derivative of z with respect to t is $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$

- If $z = f(w,x,y)$, $w = g_1(r,s)$, $x = g_2(r,s)$ and $y = g_3(r,s)$ then

  The partial derivative of z with respect to r is $\frac{\partial z}{\partial r} =$

  The partial derivative of z with respect to s is $\frac{\partial z}{\partial s} =$

- **General case** If $z = f(x_1, x_2, ..., x_n)$, $x_1 = g_1(y_1, y_2, ..., y_m)$, $x_2 = g_2(y_1, y_2, ..., y_m)$ and $x_n = g_n(y_1, y_2, ..., y_m)$ where m and n are positive integers then there are m possible partial derivatives of z which are

$$\frac{\partial z}{\partial y_1} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_1} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_1} + ... + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_1}$$

$$\frac{\partial z}{\partial y_2} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_2} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_2} + ... + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_2}$$

$$\vdots$$

$$\frac{\partial z}{\partial y_m} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_m} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_m} + ... + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_m}$$

The change in y_i affects x_i and subsequently affects z.

Ex. 17: If $w = f(x,y,z) = 3x^2y + xyz - 4y^2z^3$, where

$$x = 2r - 3s \qquad y = 6r + s \qquad z = r - s$$

Show that $\frac{\partial w}{\partial s} = x(3x - 19y + z) + yz\left(-3 - 8z^2 + 12yz\right)$.

Ex. 18: If $z = \dfrac{x+e^y}{y}$, $x = rs + se^{rt}$ and $y = 9 + rt$, find $\dfrac{\partial z}{\partial s}$ when $r = -2$, $s = 5$ and $t = 4$.

[Ans: $-2 + e^{-8}$]

Ex. 19: Determine $\dfrac{\partial y}{\partial r}$ if $y = x^2 \ln\left(x^4 + 6\right)$ and $x = (r+3s)^6$.

Ans: $12x(r+3s)^5\left[\dfrac{2x^4}{x^4+6} + \ln\left(x^4+6\right)\right]$

Ex. 20: Given that $z = e^{xy}$, $x = r - 4s$, and $y = r - s$, find $\dfrac{\partial z}{\partial r}$ in terms of *r* and *s* only.

Ans: $\dfrac{\partial z}{\partial r}(2r-5s)e^{r^2-5sr+4s^2}$