EE435 / 2 / 2020

Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

Full name Solution **Student ID.** ________

Question 1 (50 points) Your score.....................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1 - 1.5B + 0.9B^2)R_t = 0.25 + \varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean $(\mu) = 0$ and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score.....................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

The AR(2) model:

$$R_t = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + \varepsilon_t$$

we can check the stationary (weakly) by

1. Using the characteristic eq.

$(1 - 1.5B + 0.9B^2)$

$B = +1.5 \pm \quad (1.5)^2 - 4(0.9)(1)$

$$\overline{}$$
$$2(0.9)$$

roots:

$\therefore$ $0.833 + 0.645i$, $0.833 - 0.645i$

$\therefore$ $\text{Modulus}_1 = \sqrt{(0.833)^2 + (0.645)^2} = 1.054$

$\text{Modulus}_2 = \sqrt{(0.833)^2 + (-0.645)^2} = 1.054$

since modulus greater than 1 for all roots, therefore this AR(2) model is weakly stationarity.

② Using the reverse characteristic eq.

$(x^2 - 1.5x + 0.9)$ $\sqrt{-0.58^2} = \sqrt{0.58^2} \cdot \sqrt{-1}$

$$x = \frac{+1.5 \pm \sqrt{\underbrace{(1.5)^2 - 4(1)(0.9)}}}{2(0.9)}$$

roots:

$0.75 + 0.58i$, $0.75 - 0.58i$"

$$\text{Modulus}_1 = \sqrt{(0.75)^2 + (0.58)^2} = 0.949$$

$$\text{Modulus}_2 = \sqrt{(0.75)^2 + (-0.58)^2} \approx 0{,}949$$

Since modulus less than 1 for all roots , therefore this AR(2) model is weakly stationarity.

* You can use either " characteristic eq" or " reverse characteristic eq"

Question 1.2 (10 points)

Your score.......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

From

$$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$$

$\boxed{1}$ Find the unconditional mean : $E[R_t]$

$$E[R_t] = 0.25 + 1.5\,E[R_{t-1}] - 0.9\,E[R_{t-2}] + \underbrace{E[\varepsilon_t]}_{=0} \text{ from } \varepsilon_t \overset{iid}{\sim} N(0, \sigma_a^2)$$

$E[R_t] = E[R_{t-1}] = E[R_{t-2}] = \mu$ from the property of weakly stationary.

$$\therefore \mu = 0.25 + 1.5\mu - 0.9\mu$$

$$\therefore \mu = E[R_t] = \frac{0.25}{1-1.5+0.9} = 0.625$$

$\boxed{2}$ Find the conditional mean : $E[R_t | F_{t-1}]$

where $F_{t-1} = R_{t-1}, R_{t-2}, \ldots$

we know this value.

$$E[R_t|F_{t-1}] = 0.25 + 1.5E[R_{t-1}|F_{t-1}]$$

$$-0.9\underbrace{E[R_{t-2}|F_{t-2}]}_{\text{we know this value.}} + \cancel{E[\varepsilon_t|F_{t-1}]}^{\,0}$$

$$\therefore E[R_t|F_{t-1}] = 0.25 + 1.5R_{t-1} - 0.9R_{t-2}$$

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Question 1.3 (10 points) Your score......................

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

Find the unconditional variance: $Var(R_t)$

we know that,

$$(R_t-\mu) = 1.5(R_{t-1}-\mu) - 0.9(R_{t-2}-\mu) + \varepsilon_t$$

$$(R_t-\mu)^2 = (1.5)^2(R_{t-1}-\mu)^2 + (-0.9)^2(R_{t-2}-\mu) + \varepsilon_t^2$$
$$+2.(1.5)(-0.9)(R_{t-1}-\mu)(R_{t-2}-\mu)$$
$$+2.(1.5)(R_{t-1}-\mu)(\varepsilon_t) + 2.(-0.9)(R_{t-2}-\mu).(\varepsilon_t)$$

Then, take the $E(\cdot)$ to both sides,

$$E(R_t-\mu)^2 = (1.5)^2 E(R_{t-1}-\mu)^2 + (-0.9)^2 E(R_{t-2}-\mu)^2$$
$$+E(\underbrace{\varepsilon_t^2}_{\sigma_\varepsilon^2=0.25}) + 2.(1.5)(-0.9)E[\underbrace{(R_{t-1}-\mu)(R_{t-2}-\mu)}_{\gamma_1}]$$
$$+2(1.5)E[\underbrace{(R_{t-1}-\mu)(\varepsilon_t)}_{0}] + 2(-0.9)E[\underbrace{(R_{t-2}-\mu)(\varepsilon_t)}_{0}]$$

$$\Rightarrow E(R_t-\mu)^2 = E(R_{t-1}-\mu)^2 = E(R_{t-2}-\mu)^2 = Var(R_t)$$

$\because$ R_t is weakly stationary.

$\therefore$

$$Var(R_t) = 2.25\,Var(R_t) + 0.81\,Var(R_t) + 0.25$$
$$- 2.7\gamma_1$$

$$\therefore\ Var(R_t)\left[1 - 2.25 - 0.81\right] = 0.25 - 2.7\gamma_1$$

$$Var(R_t) = \frac{0.25 - 2.7\gamma_1}{-2.06}$$

we need to solve for γ_1

$\gamma_1 = 0.789\gamma_0$

We know that:

$$(R_t - \mu) = 1.5(R_{t-1} - \mu) - 0.9(R_{t-2} - \mu) + \varepsilon_t$$

$$\therefore\ (R_t - \mu)(R_{t-1} - \mu) = 1.5(R_{t-1} - \mu)(R_{t-1} - \mu)$$
$$- 0.9(R_{t-2} - \mu)(R_{t-1} - \mu)$$
$$+ \varepsilon_t(R_{t-1} - \mu)$$

Take expectation: $E(\cdot)$ both sides:

$$\underbrace{E[(R_t - \mu)(R_{t-1} - \mu)]}_{\gamma_1} = 1.5\,E[\overbrace{(R_{t-1} - \mu)(R_{t-1} - \mu)}^{\gamma_0}]$$
$$- 0.9\,\underbrace{E[(R_{t-2} - \mu)(R_{t-1} - \mu)]}_{\gamma_{-1}}$$
$$+ \underbrace{E[\varepsilon_t(R_{t-1} - \mu)]}_{=0}$$

$\therefore \quad \gamma_1 = 1.5\gamma_0 - 0.9\gamma_{-1}$ but we know that $\gamma_1 = \gamma_{-1}$

$\therefore (1+0.9)\gamma_1 = 1.5\gamma_0$

$$\frac{\gamma_1}{\gamma_0} = \rho_1 = \frac{1.5}{1+0.9} = 0.789$$

Therefore,

$$\text{var}(R_t) = \gamma_0 = \frac{0.25 - 2.7[\gamma_0 \cdot 0.789]}{-2.06}$$

$$\gamma_0 = 3.493 \ \#$$

Conditional variance: $\text{var}(R_t | \cdot)$

$\text{var}(R_t | \cdot) = \text{var}[\cancel{0.25} | \cdot]$

$$+\operatorname{var}[\cancel{1.5R_{t-1}}|\cdot] + \operatorname{var}[\cancel{-0.9R_{t-2}}|\cdot]$$

$$+\operatorname{var}[\varepsilon_t|\cdot] + 2\operatorname{cov}[\cancelto{0}{1.5R_{t-1}, \varepsilon_t}|\cdot]$$

$$+2\operatorname{cov}[\cancelto{0}{-0.9R_{t-2}}, \varepsilon_t|\cdot]$$

$$\therefore \operatorname{var}(R_t|\cdot) = \sigma_\varepsilon^2 = 0.25 \quad \#$$

Question 1.4 (10 points) | Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

From

$$(R_t-\mu)(R_{t-j}-\mu) = 1.5(R_{t-1}-\mu)(R_{t-j}-\mu) - 0.9(R_{t-2}-\mu)(R_{t-j}-\mu) + \varepsilon_t(R_{t-j}-\mu)$$

Then, we take $E(\cdot)$ both sides:

$$E[(R_t-\mu)(R_{t-j}-\mu)] = 1.5E[(R_{t-1}-\mu)(R_{t-j}-\mu)] - 0.9E[(R_{t-2}-\mu)(R_{t-j}-\mu)] + \underbrace{E[\varepsilon_t(R_{t-j}-\mu)]}_{0}$$

$$\gamma_j = 1.5\gamma_{j-1} - 0.9\gamma_{j-2} \quad \text{for } \forall j = 3,4,5,\ldots$$

$$\therefore \rho_j = 1.5\rho_{j-1} - 0.9\rho_{j-2} \quad \text{for } \forall j = 3,4,5,\ldots$$

for $j=1$

$$\gamma_1 = 1.5\gamma_0 - 0.9\gamma_{-1}$$

$$\therefore \rho_1 = 1.5 - 0.9\rho_1$$

$$\rho_1 = \frac{1.5}{1.9} = 0.789$$

for $j = 2$

$$P_2 = 1.5\underbrace{P_1}_{0.789} - 0.9\underbrace{P_0}_{=1} = 0.284$$

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Question 1.5 (10 points) | Your score......................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

From $R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$; $t = 1000$

1- step ahead forecasting

$R_{1001} = 0.25 + 1.5R_{1000} - 0.9R_{999} + \varepsilon_{1001}$

we know this 0

$E[R_{1001}|\cdot] = 0.25 + 1.5\cancel{E}[R_{1000}|\cdot] - 0.9\cancel{E}[R_{999}|\cdot] + \cancel{E}[\varepsilon_{1001}|\cdot]$

we know this

$\Downarrow$

$\hat{R}_{1000}(1) = 0.25 + 1.5R_{1000} - 0.9R_{999}$

$= 0.25 + 1.5(0.01) - 0.9(0.02) = 0.247$

1 - step ahead forecasting error :

$\therefore e_{1000}(1) = R_{1001} - \hat{R}_{1000}(1) = \varepsilon_{1001}$

$\therefore$ 1-step ahead variance :

$var(e_{1000}(1)|\cdot) = var(\varepsilon_{1000}|\cdot) = \sigma_\varepsilon^2$

$= 0.25$

∴ 95% CI for R_{1001} is

$$\hat{R}_{1000}(1) \pm 1.96\sqrt{var(e_{1000}(1))}$$

$$= 0.247 \pm 1.96\sqrt{0.25}$$

$$= [-0.733, 1.227]$$

2-step ahead forecasting:

$$R_{1002} = 0.25 + 1.5R_{1001} - 0.9R_{1000} + \varepsilon_{1002}$$

$$E[R_{1002}|\cdot] = 0.25 + 1.5\underbrace{E[R_{1001}|\cdot]}_{\text{we don't know this}} - 0.9\underbrace{E[R_{1000}|\cdot]}_{\text{we know is}} + \underbrace{E[\cancel{\varepsilon_{1002}}|\cdot]}_{=0}$$

$$= 0.25 + 1.5\hat{R}_{1000}(1) - 0.9R_{1000}$$

$$= 0.25 + 1.5[0.247] - 0.9(0.01) = 0.6115$$

2-step ahead forecasting error:

$$e_{1000}(2) = R_{1002} - \hat{R}_{1000}(2) = 1.5[R_{1001} - \hat{R}_{1000}(1)] + \varepsilon_{1002}$$

$$= 1.5\,\varepsilon_{1001} + \varepsilon_{1002}$$

$\therefore$ 2-step ahead variance :

$$\mathrm{var}(e_{1000}(2)) = (1.5)^2 \cdot \underset{\nearrow 0.25}{\sigma_\varepsilon^2} + \underset{\nearrow 0.25}{\sigma_\eta^2} = 0.8125$$

$\therefore$ 95% CI for R_{1002} is

$$\hat{R}_{1000}(2) \pm 1.96\sqrt{\mathrm{var}(e_{1000}(2))}$$

$$= 0.6115 \pm 1.96\sqrt{0.8125}$$

$$= [-1.552,\ 2.3782]$$

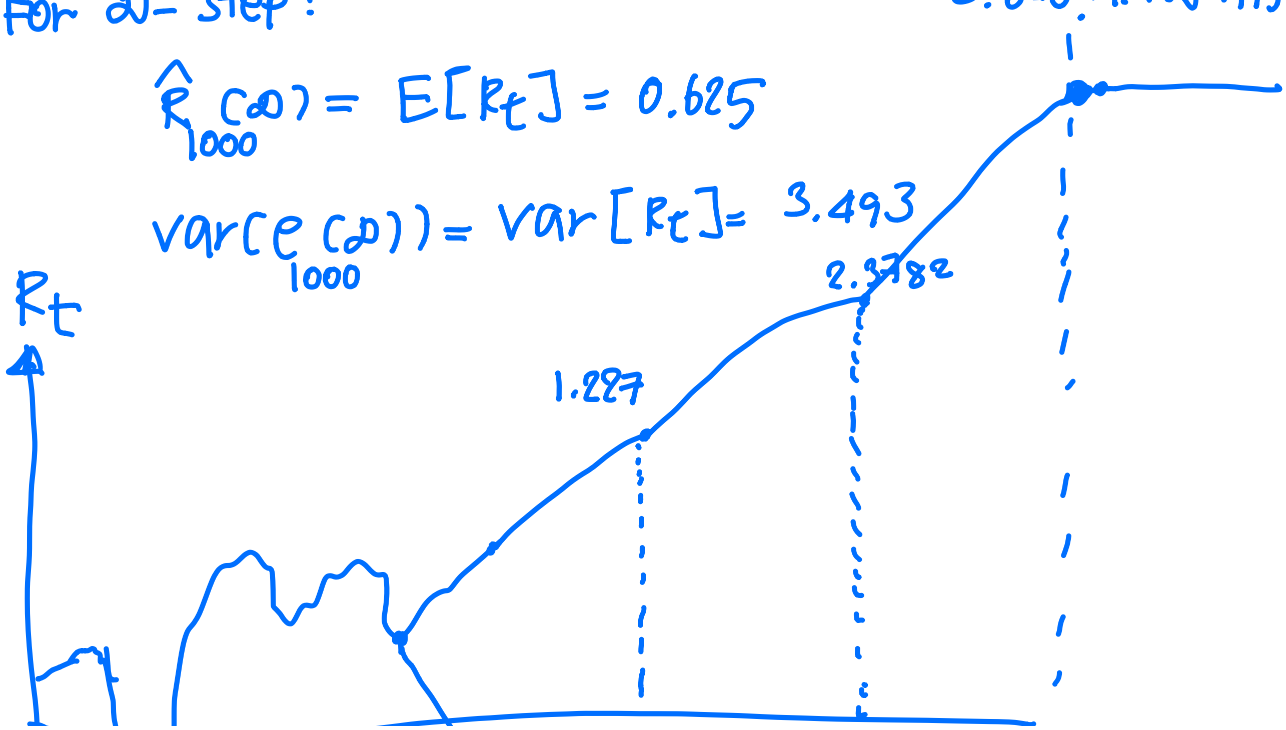
For ∞-step :

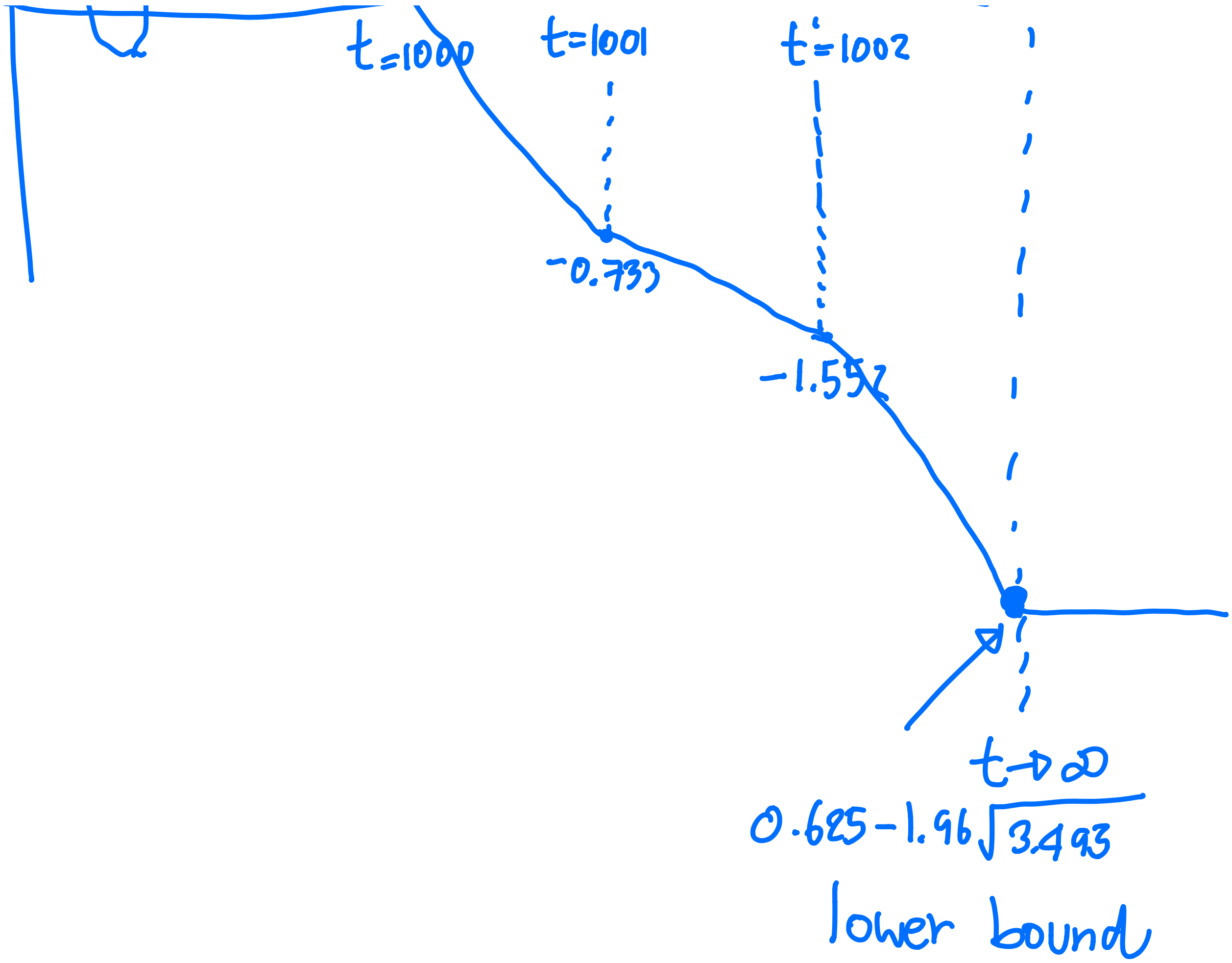
$$\hat{R}_{1000}(\infty) = E[R_t] = 0.625$$

$$\mathrm{var}(e_{1000}(\infty)) = \mathrm{var}[R_t] = 3.493$$

upper bound

$0.625 + 1.96\sqrt{3.493}$





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