Example 6.F: Consider a demand function with the form: $Q = Ap^{-0.28}m^{-0.34}$, where $p = price\ \ and\ \ m = income.$

$$\varepsilon_p^Q = \frac{\partial Q}{\partial p}\cdot\frac{p}{Q} = \left(A(-0.28)\cdot p^{-0.28-1}\, m^{-0.34}\right)\cdot\frac{p}{Q}$$

$p\uparrow 1\%$ inelastic demand f^2

$Q\downarrow 0.28\%$.

$$= \frac{\cancel{A}(-0.28)\,\cancel{p}^{-0.28-1}\cdot\cancel{m}^{-0.34}\cdot\cancel{p}}{\cancel{A}\cdot\cancel{p}^{-0.28}\cdot\cancel{m}^{-0.34}} = -0.28$$

any points on the demand curve $\rightarrow \varepsilon_p^Q$: remains constant, equal to -0.28

ε_m^Q

$$\hookrightarrow \ln Q = \ln A + \ln p^{-0.28} + \ln m^{-0.34}$$

$$\underbrace{\ln Q}_{Y} = \underbrace{\ln A}_{X_1} - 0.28\underbrace{\ln p}_{X_2} - 0.34\underbrace{\ln m}_{X_3}$$

$$\frac{\partial \ln Q}{\partial \ln m} = \underbrace{-0.34}_{\text{Inferior goods}} \qquad \varepsilon_m^Q = \frac{\partial Q}{\partial M}\cdot\frac{M}{Q} = \frac{\partial \ln Q}{\partial \ln m} = \frac{\partial y}{\partial x_3}$$

Comparative Static Analysis using partial derivative

- Model → solve for reduced-form equations of endogenous variables.
  - Endo as a function of (i) exogenous variables and (ii) parameters
  - We can apply partial derivate to the derived endogenous equilibrium solution.
  - This allows us to see how endogenous variables respond to the changes in exogenous variables and parameters.

Example 6.G: Recall that in a simple macroeconomic model, the solution for equilibrium output can be given by $y^* = \frac{I_0+G_0}{1-(1-t)b}$, where t is level of income tax and b is marginal propensity to consume. What about the effect of an increase in "b" and an increase in "t" on output?

Δ approach $\rightarrow \frac{\Delta y^*}{\Delta I} = \frac{\Delta y^*}{\Delta G_0} = \frac{1}{1-(1-t)\cdot b} > 0$

if $1-(1-t)\cdot b > 0$

delta ~~p~~approach applied to comparative static analysis.

Derivative approach

$$\frac{\partial y^*}{\partial t} = \frac{\cancel{(1-(1-t)\cdot b)(0)} - (I_0+G_0)(b)}{(1-(1-t)\cdot b)^2} < 0$$

$$\frac{\partial y^*}{\partial b} = \frac{(1-(1-t)\cdot b)(0) - (I_0+G_0)(-(1-t))}{(1-(1-t)\cdot b)^2}$$

$$= \frac{\overbrace{(I_0+G_0)}^{>0}\overbrace{(1-t)}^{>0}}{\underbrace{(1-(1-t)\cdot b)^2}_{>0}} > 0$$

$t<1$

It is more likely

$b\uparrow \rightarrow$ ~~more likely~~ ~~that~~ that households ~~household would consume more~~ would consume more.

6.2.2) Total differential

- Total differential in "y" = Total change in "y"
- Recall the concept of single variable differential
  - $dy = f'(x)dx$
  - $f'(x) = marginal\ change\ in\ \text{"y"}\ per\ a\ unit\ of\ change\ in\ x.$
  - dx = units of change in "x".
- How about total differential in multivariate relationship.

Definition: Suppose $y = f(x_1, x_2, x_3, \dots, x_n)$

$$dy = \sum_{i=1}^{N} f_i dx_i$$

Tips: *When you are taking the total differential, you are just taking all the partial derivatives and adding them up.*

$$dy = \underbrace{\underbrace{\frac{\partial f}{\partial x_1}}_{\text{marginal treatment of } x_1 \text{ only}} \cdot \underbrace{dx_i}_{\text{size of change in } x_1}}_{\text{Total change of y with respect to } x_1} + \frac{\partial f}{\partial x_2} dx_2 + \dots \frac{\partial f}{\partial x_n} \cdot dx_n$$

$$dy = \sum_{i=1}^{N} \text{Total change of y with respect to each } x_i$$

Example 6.H: Suppose $(y, r) = \sqrt{\frac{y}{r}}$. Derive total differential of C.

$$c(y, r) = \sqrt{\frac{y}{r}} \Rightarrow \underline{dc}$$

dc → change in y

dc → change in r

$$dc = \underbrace{\frac{\partial c}{\partial y} \cdot dy}_{\text{attribution of the change in y would impact the consumption}} + \underbrace{\frac{\partial c}{\partial r} \cdot dr}_{\text{attribution of the change in r that would impact the consumption}}$$

$$c = \sqrt{\frac{y}{r}}$$

$$dc \; \cancel{\partial c} = \boxed{\frac{1}{2} \cdot \frac{y^{-1/2}}{r^{1/2}} \cdot dy + -\frac{1}{2} \cdot \frac{y^{\frac{1}{2}}}{r^{3/2}} \cdot dr}$$

~~To change~~

Total change in Consumption = (per unit change in c with respect y)(units of change in y) + (per unit change in c with respect r)(units of change in r)

Example 6.J: Consider the equilibrium function of real income $y^* = f(I_0, G_o, t, b) = \frac{I_0+G_0}{1-(1-t)b}$. Find total differential of y^*.

$$\frac{\partial y^*}{\partial I_0}, \frac{\partial y^*}{\partial G_0}, \frac{\partial y^*}{\partial t}, \frac{\partial y^*}{\partial b}$$

$dy^* = \Sigma$ total change of y^* with respect to each of the variables that affects the level of y^*.

$$dy^* = \frac{\partial y^*}{\partial I_0} \cdot dI_0 + \frac{\partial y^*}{\partial G_0} \cdot dG_0 + \frac{\partial y^*}{\partial t} \cdot dt + \frac{\partial y^*}{\partial b} \cdot db$$

$$dy^* = \frac{1}{1-(1-t)\cdot b} \cdot dI_0 + \frac{1}{1-(1-t)\cdot b} \cdot dG_0 + \frac{-(I_0+G_0)\cdot b}{(1-(1-t)\cdot b)^2} \cdot dt + \frac{(I_0+G_0)(1-t)}{(1-(1-t)\cdot b)^2} \cdot db$$

6.2.3) *Total derivative (aka. Chain rule!)*

- For an illustrative purpose, consider the consumption function in **example 6.H.** Suppose that *"y" and "r" grows over time.*
- That is, $y = g(t)$ and $r = h(t)$ where t is the numbers of period from now.
- By the composite function, consumption will be *ultimately* determined by "t".
- Then, how do we calculate the derivative of "c" with respect to "t"?
- This requires the concept of total derivative.

$\frac{dc}{dt}$

ultimate factor that drives consumption

- Total differential in "c" can be given by

$$dC = \frac{\partial c}{\partial y}dy + \frac{\partial c}{\partial r}dr$$

- Given this assumption, one obtains that

$y = g(t) \longrightarrow$
$$dy = g'(t)dt$$

$r = h(t) \longrightarrow$
$$dr = h'(t)dt$$

- Applying the two expressions above, we obtain that

$$dC = \frac{\partial c}{\partial y}g'(t)dt + \frac{\partial c}{\partial r}h'(t)dt$$

- As a result, total derivative of C with respect to t is given by

$$\frac{dC}{dt} = \frac{\partial c}{\partial y}g'(t) + \frac{\partial c}{\partial r}h'(t)$$

final outcome of total derivative.

Example 6.K Consider a consumption function with $C = t^2 + \sqrt{y} - r^3$ and $y = 2t^3$ and $r = t + 1$. Derive the total derivative of C with respect to t.

$C(t, y, r)$

$$dC = \frac{\partial C}{\partial t}.dt + \underbrace{\frac{\partial C}{\partial y}.dy + \frac{\partial C}{\partial r}.dr}$$

direct impact

Indirect channels through which time variation can affect consumption.

$C = t^2 + \sqrt{y} - r^3$

$y = 2t^3$; $r = t + 1$

$$dC = 2t.dt + \frac{1}{2\sqrt{y}}.dy - 3r^2.dr$$

$6t^2 dt$ $\quad$ $1.dt$

$$dC = 2.t.dt + \frac{1}{2\sqrt{y}}.6t^2.dt - 1(3r^2).dt$$

$$\left.\frac{dC}{dt}\right|_{t=1} = \left(2t + \frac{6t^2}{2\sqrt{y}} - 3r^2\right) = 2 + \frac{6}{2\sqrt{2}} - 3(2)^2$$

< 0

find the interval "t" such that $\frac{dC}{dt}$ is Increasing.

$c = c(y, r) \Rightarrow c = c(y(t,s), r(t,s))$
$= \hat{c}(t, s)$ $\quad \frac{\partial c}{\partial t}$; $\frac{\partial c}{\partial s}$

6.2.4) Partial total derivative (aka. Chain rule for multivariate!)

- From the example above, what if both y and r depend on "t" and "s"?
- Consumption would then be ultimately driven by "t" and "s".
- The concept is changed to *partial total derivative*: $\frac{\partial c}{\partial t}$ *and* $\frac{\partial c}{\partial s}$

Following the same method as we applied to total derivative, we know that

$dc = \partial c/\partial y \cdot dy + \frac{\partial c}{\partial r} \cdot dr$

$$dC = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial s}ds\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial s}ds\right)$$

As a result, $ds = 0$ (assuming)

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial t}\right)$$

partial total derivative of c w.r.t. respect to t

Meanwhile, we would yield that

$(dt = 0) \rightarrow t$ is fixed

$$\frac{\partial c}{\partial s} = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial s}\right)$$ $\leftarrow$ partial total derivative of c w.r.t. respect to s.

6.3) The Implicit derivative

Derive

- Derivative method applied to an ***implicit function***

Explicit function	Implicit function
• $y = 2x - 1$ • $y = x^2 + 2x + 1$ • $y = \ln(2x+1) + 2x + 1$ • $z = \ln(2x+1) + 2y^2x$	• $x^2 + y^2 = 3$ • $xw + yw - w^2 + zx^2 = 9$

For the implicit function, how do we obtain the derivative of y with respect to x?

Method 1: (brute force) → Rewrite y in terms of x.

- Comments:
  - Tedious… and NOT applicable for all the cases.
  - In many cases, the function doesn't admit any closed-form solutions: impossible to rewrite "y" in term of "x".

Example 6.L Suppose that $x^2 + y^2 = 3$, find dy/dx.

Brute approach (~~…~~)
... approach

$y = \pm\sqrt{3-x^2}$

$\frac{dy}{dx} = \pm \frac{1}{\cancel{2}\sqrt{3-x^2}}(-\cancel{2}x)$

$= -\left(\pm \frac{x}{\sqrt{3-x^2}}\right) = -\frac{x}{y}$

Implicit function approach:

$F(x,y) = x^2 + y^2 - 3 = 0$

$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y}$

$= -\frac{x}{y}$ #

$\left.\frac{dy}{dx}\right|_{x=0} = 0$

$\left.dy/dx\right|_{x=1} = -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Method 2: Apply the implicit function theorem

Theorem: Suppose a given function is written in the form of $F(x, y) = 0$,

$$\longrightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proof:

$F(x,y) = 0$

$dF = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy = 0$

$\frac{dy}{dx}$

$\frac{\partial F}{\partial y} \cdot dy = -\frac{\partial F}{\partial x} \cdot dx$

$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\partial F / \partial y} = -\frac{F_x}{F_y}$

Example 6.M Suppose that $\ln(x+2) - y^2 - e^{x+y} = 3$, *Find* $\frac{dy}{dx}$

Not doable if you don't know implicit f' Theorem:

$$F(x,y) = \ln(x+2) - y^2 - e^{x+y} - 3 = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{\frac{1}{x+2}(1) - \cancel{2y} - e^{x+y}(1)}{-2y - e^{x+y}(1)}$$

$$= \frac{\frac{1}{x+2} - e^{x+y}}{e^{x+y} + 2y}$$

$dy/dx \,|_{y=0}$

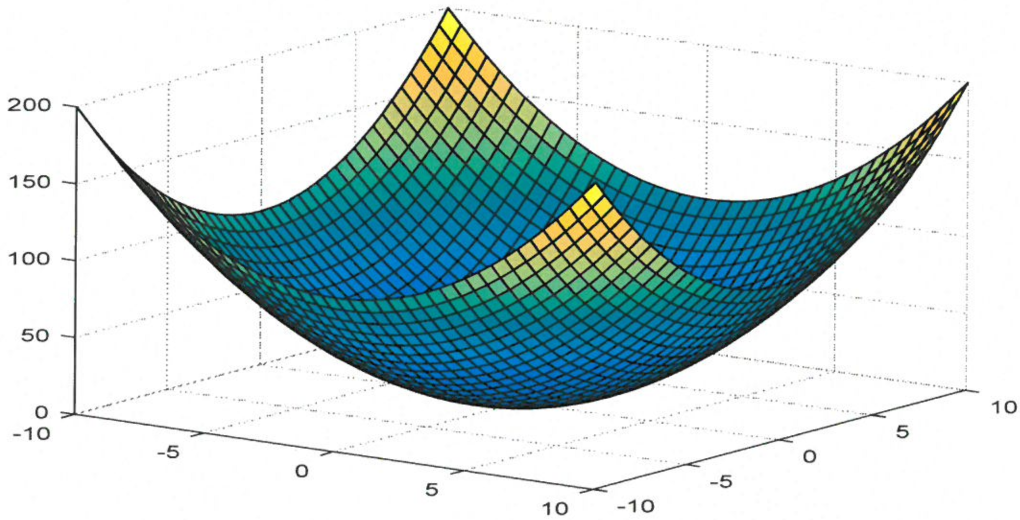
Level set (Contour set)

Definition: Suppose a function $z = f(x, y)$, the level set of "f", denoted by $L(f, z_0)$, is defined as

$$L(f, z_0) = \{(x, y) |\, f(x, y) = z_0\}$$

All possible combinations of (x,y) that generate f(x,y) = z_0.

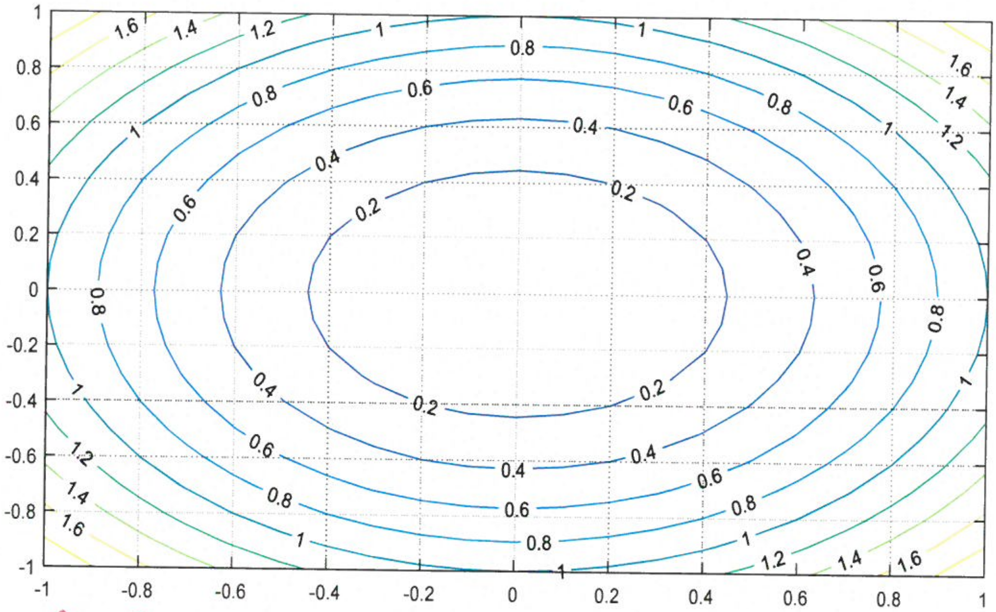
- For example, $z = f(x, y) = x^2 + y^2$



Assuming that $z_0 = 1$, $L(f, 1) = \{(x, y)|x^2 + y^2 = 1\}$

Graphically, $L(f, 1)$ can be represented in a 2-D figure. The shape of the relationship is a circle with unit radius.

Contour of level sets: By varying the value of "z", we yield several contour of level sets.



$z = x^2 + y^2 \longrightarrow \dfrac{dy}{dx}\Big|_{z=z_0} = -\dfrac{f_x}{f_y} = -\dfrac{2x}{2y} = -\dfrac{x}{y}$

Derivative of the level set

- One can apply the second result of the implicit function theorem.

Indifference Curve "

Isoquant "

$$\frac{dy}{dx}|_{z=z_0} = -\frac{x}{y}, \qquad i.e.\ \ dz = 0$$

$z = f(x, y)$

$z = z_0$

$\cancel{dz}^{\,0} = f_x\,dx + f_y \cdot dy$

z is fixed $= z_0$

$\dfrac{dy}{dx}\Big|_{z=z_0} = -\dfrac{f_x}{f_y}$

Economic example of the level set

Example 6.0: Suppose that U is the utility function of a consumer. The function takes the following form: $U = U(x,y) = xy^2$ where x is the amount of consumption on good "x" and y is the amount of consumption on good "y".

a) Derive the level set of the utility function. What do we call the level set of utility function in economics?

$L(u,\bar{u}) = \{(x,y) \mid xy^2 = \bar{U}\}$

↳ Indifference Curve

"Combinations of "(x and y" that guarantee consumer's level of satisfaction equal to $\bar{u}$.

b) Calculate slope of the level set. What does the slope mean to you in economics? $u(x,y)$; $u_x > 0$ $u_y > 0$; $u_{xx}; u_{yy} < 0$

$\cancel{x}$: $xy^2 = \bar{u}$ $\quad -\frac{MU_x}{MU_y}$ $\quad \frac{\partial u}{\partial x} = \text{MU of good } x$

$$\frac{dy}{dx}\bigg|_{u=\bar{u}} = -\frac{u_x}{u_y} = -\frac{y^2}{2xy} = -\frac{y}{2x} < 0$$

$\frac{\partial u}{\partial y} = \text{MU of good } y$

slope of IC

slope < 0

MRS of x for y → $\frac{MU_x}{MU_y} = |\text{slope}|_{IC}$

MRS_{xy} ↑ or ↓ if x ↑ ~~more~~ more

$\boxed{\text{decrease}}$

~~x would be so valuable to you~~

low consumptn on x   MRS high

high consumptn on x   MRS low

x would be so valuable to you.
priceless

x would be less valuable to you.
you are willing to sacrifice y for x
less than before.

$$\frac{dMRS}{dx} < 0$$

MRS: trade-off rate of good x for good y in order to keep $u = \bar{u}$ (fixed)

$U = u(x, y)$

we can show that $\frac{dMRS}{dx} < 0$

$MRS = \left|\frac{dy}{dx}\right| = \text{abs (slope of IC)}$

$\text{slope of IC} = -\frac{u_x}{u_y} = -\frac{MU_x}{MU_y}$

$MU_x = y^2$

$MU_y = 2xy$

$\frac{dMRS}{dx} = \frac{d\left(\frac{y}{2x}\right)}{dx} = -\frac{1}{2} \cdot \frac{y}{x^2}$ Partially correct

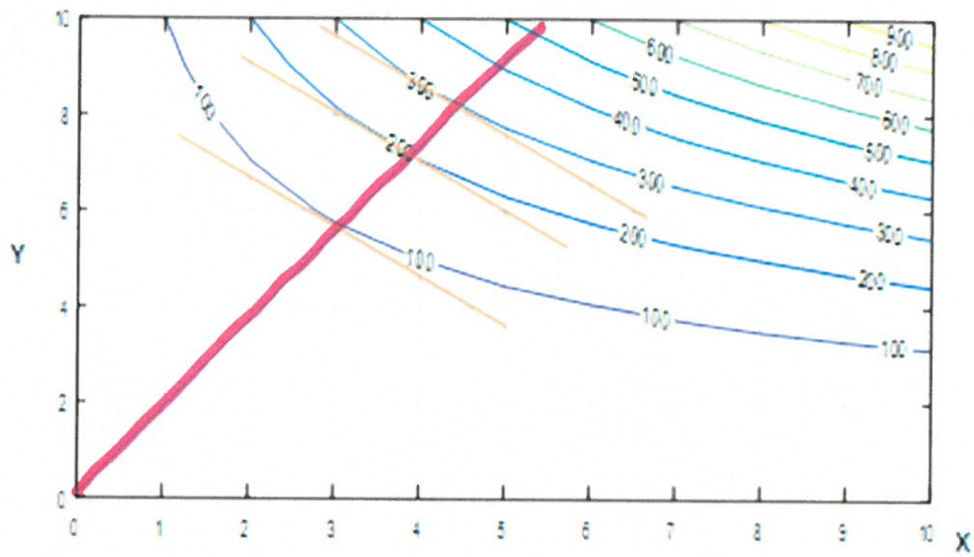
→ y is not fixed. (adjusted along dy/dx)

$\Rightarrow \frac{\frac{1}{2}\left[x \cdot \overbrace{\frac{dy}{dx}\Big|_{u=\bar{u}}}^{\left(-\frac{y}{2x}\right)} - y \cdot \overbrace{\frac{dx}{dx}}^{1}\right]}{x^2} = \frac{\frac{1}{2}\left(-\frac{y}{2} - y\right)}{x^2}$

$\Rightarrow -\frac{3}{4} \cdot \frac{y}{x^2} < 0$

Locus of indifference curves

y/x : fixed



homothetic utility f^{n}.

for the same $\frac{y}{x}$; MRS is the same.

6.3.1) Implicit function and its Application for comparative static analysis

a) Comparative static analysis without explicit solution

Motivating example: Consider a market model where market demand and market supply are given by,

$$Q^d = D(P, y) = a - bP + cy,$$

$$Q^s = S(P) = e + fp \quad S_p > 0$$

Comparative static analysis: ***how does the level of income affect equilibrium output and price?***

Approach for the analysis:

- Mechanically, we solve for the solution (reduced-form solution)
- Apply derivative technique, and check for the sign of derivative.
- Then reach the conclusion over the impact of income (exogenous changes) on equilibrium output and price (the two endogenous variables in our model.)

These are the procedure that we did for the midterm exam. Then, what's new?

Motivating question: would the conclusion be obtained under a wider class of the model equations? (e.g. non-linear function) Could it be shown mathematically? $\frac{\partial Q^*}{\partial y}$, $\frac{\partial P^*}{\partial y}$ =???

Why is this important? → Economics usually want to get some testable implications under more ***robust*** assumptions.

- People might find it less convincing to assume linear demand, and hence arguing that your conclusion is only true under the too restricted assumptions.

- But, people might be more comfortable with the assumption that law of demand and law of supply holds ($D_p < 0$; $S_p < 0$), together with assuming that the production in question is normal goods ($D_y > 0$).

Implicit derivative can be applied into this context. How?

Note first:

$$S(P) = D(P, y) \rightarrow \frac{\partial P}{\partial y}$$

$S(P) - D(P, y) = 0$

$F(y, x) = 0 \rightarrow$ Implicit derivative // $= -\frac{F_x}{F_y}$

Under some conditions, we should be able to write P in terms of Y as the explicit function, i.e. deriving the reduced-form equation.

Don't need to write p in terms of y. even if we could.

$F(P,y)$ $S(p) - D(p,y) = 0 \Rightarrow$ Implicit f^n

$F(y,x) = 0$

$$\frac{dp}{dy} = -\frac{F_y}{F_p} = -\frac{-\frac{\partial D}{\partial y}}{\frac{\partial S}{\partial p} - \frac{\partial D}{\partial p}}$$

$+ \Rightarrow$ normal good

$dP/dy > 0$

$\rightarrow P\uparrow$

$$= \frac{\frac{\partial D}{\partial y}}{\underbrace{\underset{+}{\frac{\partial S}{\partial p}} - \underset{-}{\frac{\partial D}{\partial p}}}_{+}}$$

$- \Rightarrow$ Inferior good

$\frac{dp}{dy} < 0$

$y\uparrow; \rightarrow P\downarrow$

$P = 0$

$Q^d = D(P; t)$ tax rate $\quad D_t < 0$

$Q^s = S(P; w)$ $\quad S_w < 0$ wage rate

Calculate $\frac{dp^*}{dt} < 0$ and $\frac{dp^*}{dw} > 0$

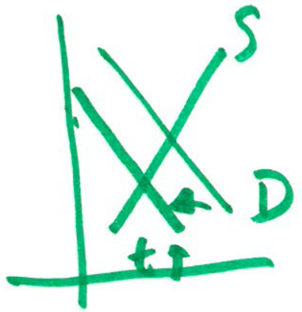
$\frac{dQ^*}{dt}$ and $\frac{dQ^*}{dw}$

$Q^d = Q^s \rightarrow D(P, t) = S(P, w)$

$F(P; t, w) = D(P, t) - S(P, w) = 0$

$F_P = D_p - S_p$

$$\frac{dp}{dt} = -\frac{F_t}{F_p} = -\frac{D_t}{D_p - S_p} = -\frac{\overset{<0}{D_t}}{\underbrace{\underset{-}{D_p} - \underset{+}{S_p}}_{\ominus}} \quad \ominus$$



$$\frac{dP}{dw} = -\frac{F_w}{F_p} = -\frac{-S_w}{D_p - S_p} = \left.\frac{S_w \quad \ominus}{\underbrace{\underset{-}{D_p} - \underset{+}{S_p}}_{\ominus}}\right\} \underline{\underline{>0}}$$

$$Q = S(P, w)$$

$$Q = D(P, t)$$

$$\frac{dQ}{dw} = \underbrace{\overset{\oplus}{\frac{ds}{dp}} \cdot \overset{\oplus}{\frac{dP}{dw}}}_{\text{indirect impact}} + \underbrace{\overset{\ominus}{\frac{ds}{dw}}}_{\text{direct impact.}}$$

$$S_p \cdot \frac{dp}{dw} + S_w$$

$$= \frac{S_w}{D_p - S_p}$$

$$\frac{S_p \cdot S_w}{D_p - S_p} + S_w = \frac{S_p \cdot S_w + S_w D_p - S_w S_p}{D_p - S_p}$$

$$= \frac{\overset{-}{S_w} \cdot \overset{-}{D_p}}{\underset{-}{D_p} - \underset{+}{S_p}} \left.\begin{matrix} \} \oplus \\ \} \ominus \end{matrix}\right\} \ominus$$

b) Comparative static in an aspect of optimization

Consider a profit-maximization problem $\pi(L) = pf(L) - wL$ where $f(L)$ is increasing and concave in L.

Following the first-order condition:

$$\frac{\partial \pi}{\partial L} = 0 \rightarrow \text{optimal labor employment}$$

$$\underbrace{p \cdot f'(L) - w = 0}_{\text{Implicit } f^{n}.}$$

By applying the implicit function, we yield that

$$F(L; p, w) = p \cdot f'(L) - w \; ; \; F_L = p \cdot f''(L)$$

$$\frac{dL}{dp} = -\frac{F_p}{F_L} = -\frac{F_p}{p \cdot f''} = -f'(L)/pf'' = \left.-\frac{f'}{pf''}\right\} (+)$$

$$\frac{dL}{dw} = -\frac{F_w}{F_{L}} = -\frac{F_w}{pf''} = -(-1/pf'') = \left.\frac{1}{pf''}\right\} (-)$$