EE320 (2/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION WITHOUT CONSTRAINTS:

MORE-THAN-ONE-INDEPENDENT VARIABLE CASES

Topics

- The differential version of optimization condition: One variable case
- Two choice variable optimization
  - Conditions for maximum or minimum
  - Economics examples
- Multivariable optimization
  - Conditions for maximum or minimum
  - Economics examples

Differential Version of Optimization Conditions: One Variable Case

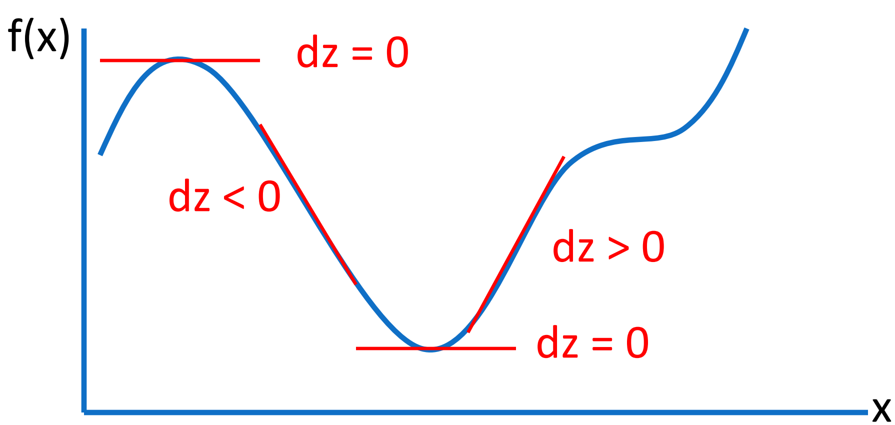
- Given $z = f(x)$, the differential of z is:

  $dz = f'(x)dx$, where $f'(x) = dz/dx$.

- The first-order condition (in terms of derivative) for an optimum:

  $f'(x) = 0$

- Graph



➢ The first-order differential condition for any arbitrary $dx \neq 0$:

$dz = 0$.

Differential Version of Optimization Conditions: One Variable Case

- The second-order sufficient condition (in terms of derivative):
  - For a maximum of z: $f''(x) < 0$.
  - For a minimum of z: $f''(x) > 0$.
- Rewrite the S.O.C. in terms of differential:

$$d^2z = d(dz) = d[f'(x)dx] = [df'(x)]dx = [f''(x)dx]dx$$

$$\rightarrow d^2z = f''(x)dx^2$$

- The second-order differential condition:
  - For a maximum of z: $d^2z < 0$.
  - For a minimum of z: $d^2z > 0$.

Example: 1-variable optimization

- Example: $y = 10x - x^2$

TWO CHOICE VARIABLE OPTIMIZATION

First-Order Condition

- For $z = f(x, y)$, the first-order necessary condition is:

  $dz = 0$ for arbitrary values of dx and dy, not both zero.

- The total differential is:

  $$dz = f_x dx + f_y dy$$

  $$dz = 0 \Rightarrow f_x = f_y = 0.$$

Theorem: A differentiable function $z = f(x, y)$ can only have a maximum or minimum at an interior point (x_0, y_0) if it is a stationary point. That is, if the point $(x, y) = (x_0, y_0)$ satisfies the two F.O.C. equations:

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

Example: F.O.C. for Two-Variable Case

Example: $z = f(x, y) = 10x + 10y + xy - x^2 - y^2$

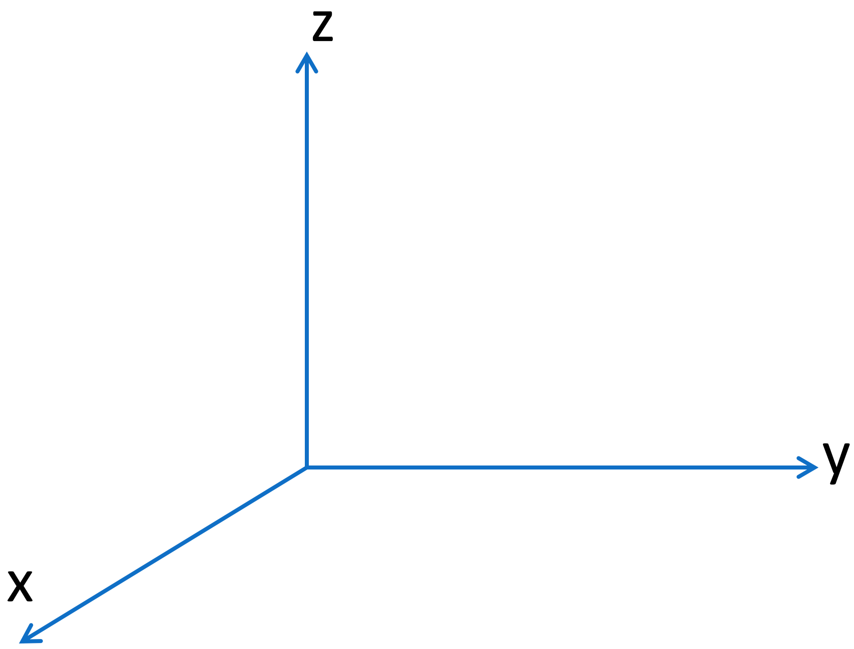
- F.O.C.'s are:

- Stationary points are:

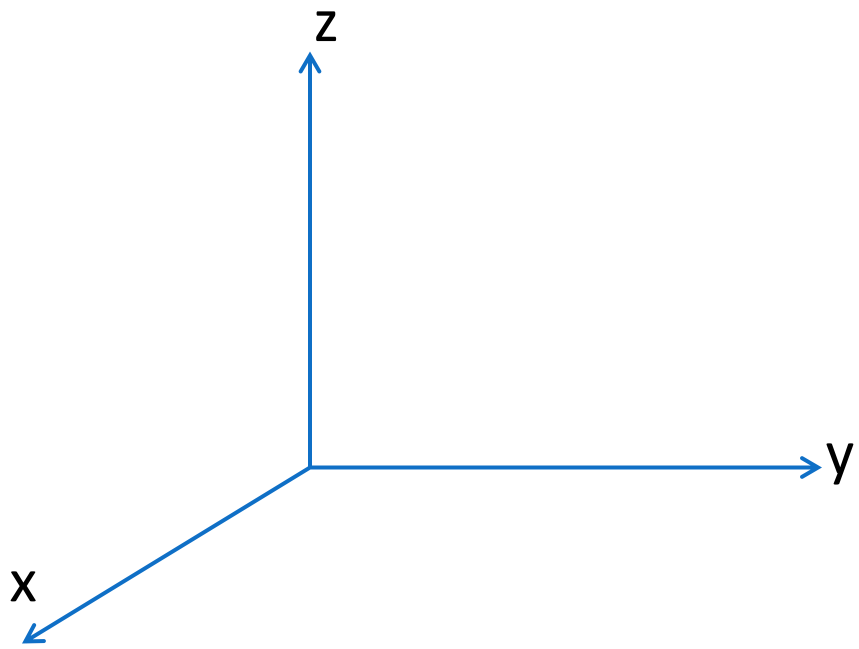
Extreme Values of a Function of Two Variables:

$z = f(x, y)$

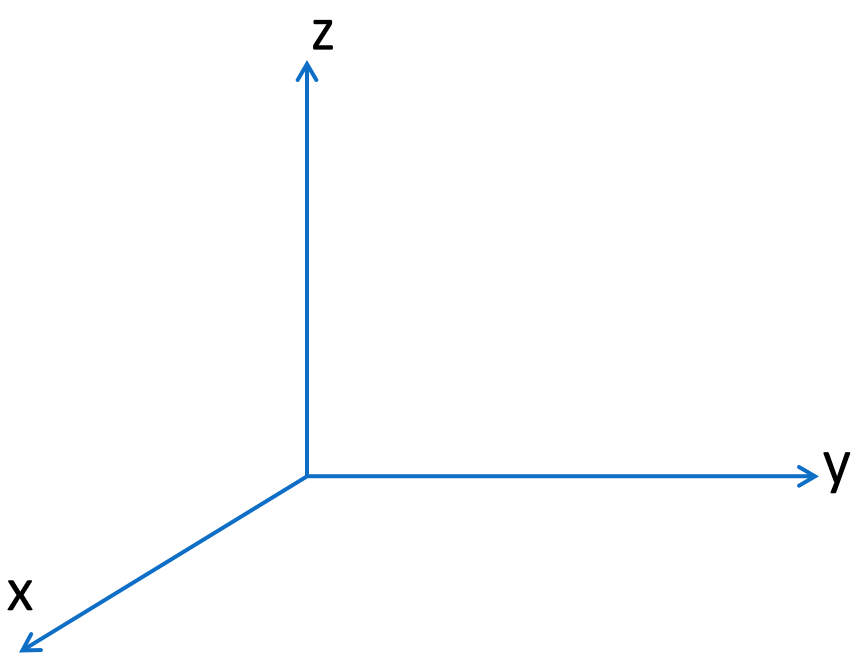
<u>Maximum value</u>

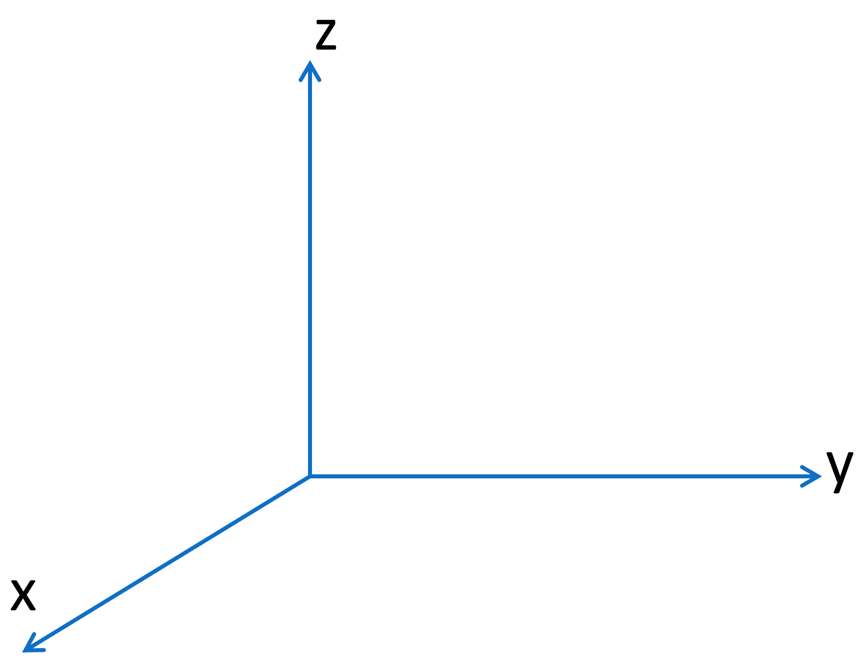


<u>Minimum value</u>



The FOC is necessary but *not sufficient* condition.





Second-Order Conditions

- To test for a maximum or a minimum, we need to check the second-order partial derivatives.
- Second-Order Total Differentials:

$$d^2z = d(dz)$$

Hessian Matrix of Second Partials

- The quadratic form $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$ can be written in matrix form as:

- The matrix in which the second-order partial derivatives are the elements is the Hessian matrix, and its determinant is called a "Hessian determinant".

- First leading principal minor: $|H_1|$ =
- Second leading principal minor: $|H_2|$ =

Rules for Maximum and Minimum: 2-Variable Case*

- Given that the first-order necessary condition is satisfied and , the second-order sufficient condition for z = f(x, y) is

1. Maximum:

   $d^2z < 0$ iff $f_{xx} < 0$, $f_{yy} < 0$, and $f_{xx} f_{yy} > (f_{xy})^2$

2. Minimum:

   $d^2z > 0$ iff $f_{xx} > 0$, $f_{yy} > 0$, and $f_{xx} f_{yy} > (f_{xy})^2$

3. Otherwise, we have a saddle point.

Note: The second-order partial derivatives are to be evaluated at the stationary point where $f_x = f_y = 0$.

*See additional notes on the restrictions for the signs of f_{xx}, f_{xy} , and f_{yy}.

Summary: Necessary and Sufficient Conditions for a Maximum and a Minimum

Given that z = f(x, y),

Condition	Maximum	Minimum
First-order necessary	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient	$f_{xx} < 0$, $f_{yy} < 0$, *and* $f_{xx}f_{yy} > (f_{xy})^2$	$f_{xx} > 0$, $f_{yy} > 0$, *and* $f_{xx}f_{yy} > (f_{xy})^2$

Example: Second-Order Condition

- $z = f(x, y) = 10x + 10y + xy - x^2 - y^2$

FOC:

SOC:

- $f_{xx} =$
- $f_{xy} =$
- $f_{yy} =$
- $f_{yx} =$

➔ d^2z

Example: 2-choice variable optimization

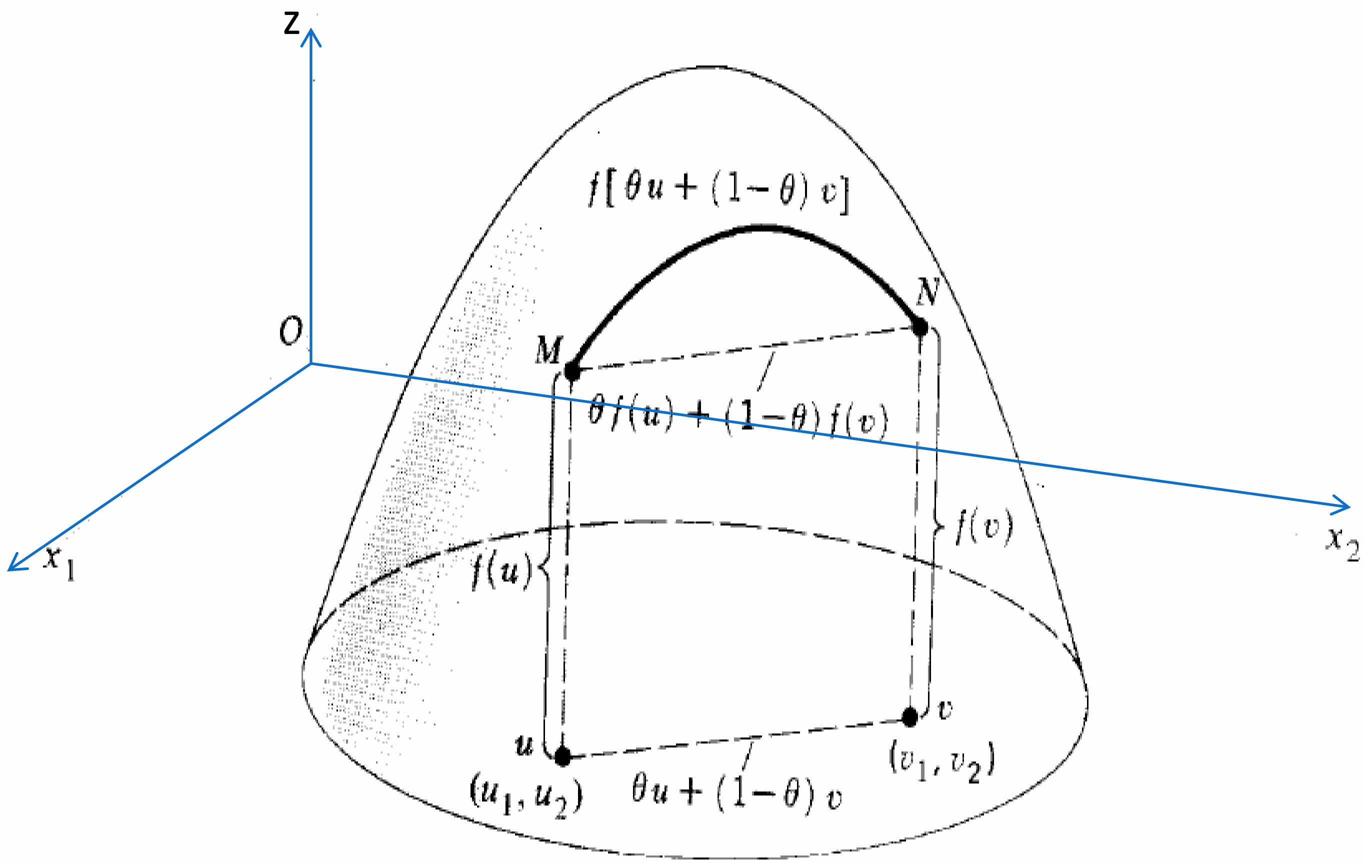
- $z = f(x, y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158.$

Concavity and Convexity: 2 variable case

- Consider a two-variable function $z = f(x_1, x_2)$.
- **Definition**:
  - The function is *concave* (*strictly concave*) if and only if, for any pair of distinct points M and N on its graph – surface – line segment MN lies either *on* or *below* (*entirely below*) the surface.
  - The function *convex* (*strictly convex*) if and only if, for any pair of distinct points M and N on its graph – surface – line segment MN lies either *on or above* (*entirely above*) the surface.

θ

Strictly Concave Function



A function *f* is strictly concave (convex) *iff*, for any pair of distinct points *u* and *v* in the domain of f, and for 0 < θ <1,

$$\theta f(u) + (1-\theta)f(v) \;<\; (>) \;\; f[\theta(u) + (1-\theta)(v)]$$

Height of the line MN Height of arc

Checking Concavity and Convexity by the Derivative Conditions

- Consider a function $z = f(x_1, x_2)$, which is twice continuously differentiable. For such a function, second-order partial derivatives exist, and thus d^2z is defined.
- Concavity and convexity can then be checked by the sign of d^2z:
  - $z = f(x_1, x_2)$ is *strictly* *concave* if, and only if, d^2z is everywhere *definite negative* ($d^2z < 0$).

    ➔ z* is a maximum.
  - $z = f(x_1, x_2)$ is *strictly* *convex* if, and only if, d^2z is everywhere *definite positive* ($d^2z > 0$).

    ➔ z* is a minimum.

where z* is the value of function z evaluated at the stationary point where $f_x = f_y = 0$.

Examples: Convexity and Concavity

Check the following functions for concavity and convexity by the derivative conditions.

- Example 1: $z = 2x^2 - xy + y^2$
- Example 2: $z = x_1^2 + x_2^2$

Application: Duopoly

- There are two firms, with identical cost: $TC_i = cQ_i$, $i = 1, 2$, and market demand is $P = a - bQ$, $Q = Q_1 + Q_2$.
- Find Q_1, Q_2 that maximizes each firm's profit.

Application: Multiproduct Firm (1)

- Competitive firm

  Let p_1 = 6, p_2 = 9 and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

  Find Q_1 and Q_2 that maximize the firm's profit.

Application : Multiproduct Firm (2)

- Monopolist

  Let $p_1 = 35 - Q_1$, $p_2 = 33 - Q_2$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

  Find Q_1 and Q_2 that maximize the firm's profit.

Application : Multiplant Firm

- Let P = 25, $TC_1 = 2Q_1^2 + 5Q_1 + 10$, and $TC_2 = 2Q_2^2 + 3Q_2 + 15$.
  Find Q_1 and Q_2 that maximize the firm's profit.

Application :
Multimarket Monopoly (Price Discrimination)

- Let $R = R_1(Q_1) + R_2(Q_2)$ and $C = C(Q)$ where $Q = Q_1 + Q_2$. Find the FONC and SOSC for profit maximization.

Application : Price Discrimination

- Let $p_1 = 22 - 2Q_1$, $p_2 = 10 – 0.5Q_2$ and TC = 2Q + 5.
  Find Q_1, Q_2, p_1, and p_2 that maximize the firm's profit.

➢ Max $\pi = (22 – 2Q_1)Q_1 + (10 – 0.5Q_2)Q_2 – [2(Q_1+Q_2)+ 5]$

➢ FONC: $\pi_1 = 22 – 4Q_1 – 2 = 0$

$\pi_2 = 10 –Q_2 – 2 = 0$

➔ $Q^*_1 = 5$, $Q^*_1 = 8$

➢ SOSC: $\pi_{11} = -4$, $\pi_{22} = -1$, $\pi_{12} = 0$

$|H_1| < 0$ & $|H| = [(4)(1)-0] > 0$

➔ SOSC for max are satisfied.

Application : Input Decision of a firm

- Let $Q = f(K, L) = 5K^{0.5}L^{0.25}$, $P = 4$, $w = 10$, $r = 5$. Find K^* and L^* that maximize profit.

Comparative-Static Aspects of Optimization

Example: Consider a two-product firm in perfect competition.
The firm's revenue is $R = P_{10}Q_1 + P_{20}Q_2$ (P_{10} and P_{20} are exogenous)
And the cost function is $C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

→Reduced-form solutions are the optimal output levels expressed in terms of exogenous variables (P_{10} and P_{20}):

$$Q_1^* = \frac{4P_{10} - P_{20}}{15} \quad \text{and} \quad Q_2^* = \frac{4P_{20} - P_{10}}{15}$$

→ Comparative-statics of the model:

$$\frac{\partial Q_1^*}{\partial P_{10}} = \frac{4}{15} \;;\; \frac{\partial Q_1^*}{\partial P_{20}} = -\frac{1}{15} \;;\; \frac{\partial Q_2^*}{\partial P_{10}} = -\frac{1}{15} \;;\; \frac{\partial Q_2^*}{\partial P_{20}} = \frac{4}{15}$$

Comparative-Static Aspects of Optimization

- General-Function Models

Consider the input-decision problem. Given that $R = P \cdot Q(K, L)$ and $C = wL + rK$, where P, w, and r are exogenous, find dL^*/dP and dK^*/dP.

MULTIVARIABLE OPTIMIZATION

3-Variables Optimization

- Let $z = f(x_1, x_2, x_3)$. Total differential is:

  dz =

➢ First-order necessary condition for an extremum of z is:

- Second-order total differential:

  d^2z =

→ Hessian matrix:

3-Variables Optimization

- The leading principal minors of the Hessian matrix are:

- Thus, second-order sufficient condition for an extremum of z:

1. z^* is a *minimum* if $|H_1| > 0$; $|H_2| > 0$; $|H_3| > 0$
   (i.e. d^2z is *positive definite*.)

2. z^* is a *maximum* if $|H_1| < 0$; $|H_2| > 0$; $|H_3| < 0$
   (i.e. d^2z is *negative definite*.)

Note: All the leading principal minors are evaluated at the stationary points where $f_1 = f_2 = f_3 = 0$.

n-variable Optimization

- Let z = f(x_1, x_2, …, x_n).
- Total differential:

➢ F.O.N.C.:

➢ S.O.S.C:

Hessian matrix: $$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

where $|H_1|$, $|H_2|$, …, $|H_n|$ are the leading principal minors.

- For a *maximum* in z, all the principal minors *alternate is sign, the first one being negative*:
- For a *minimum* in z, all the principal minors must be *positive*:

Necessary and Sufficient Conditions for a Maximum and Minimum (*n*- choice variables)

Condition	Maximum	Minimum
First-order necessary	$f_1 = f_2 = \ldots = f_n = 0$	$f_1 = f_2 = \ldots = f_n = 0$
Second-order sufficient	$\lvert H_1\rvert < 0$, $\lvert H_2\rvert > 0$, $\lvert H_3\rvert < 0$, … or $(-1)^i\lvert H_i\rvert > 0$ for i = 1, 2, …, n	$\lvert H_1\rvert > 0$, $\lvert H_2\rvert > 0$, $\lvert H_3\rvert > 0$, … Or $\lvert H_i\rvert > 0$ for i = 1, 2, …, n

Application: Multimarket Monopoly

- Let $p_1 = 63 - 4Q_1$, $p_2 = 105 – 5Q_2$, $p_3 = 75 – 6Q_3$ and $TC = 15Q + 20$. Find Q_i and p_i , for i = 1, 2,3, that maximize the firm’s profit.