

# EE320 (1/2017)

## INTRODUCTORY MATHEMATICAL ECONOMICS

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### OPTIMIZATION UNDER EQUALITY CONSTRAINTS

# Topics

- Effects of a constraint
- Finding the stationary (optimal) values
  - Elimination method
  - Lagrange multiplier
  - $n$ -variable and multi-constraint cases
- Second-order conditions for constrained optimization
- Economic applications
- Extensions:  $n$ -variable and multiconstraint cases

# Effects of a Constraint

- In the previous topic, we studied optimization problems, where the choice variables can be chosen freely.
- Example: A profit-maximizing firm's objective is:

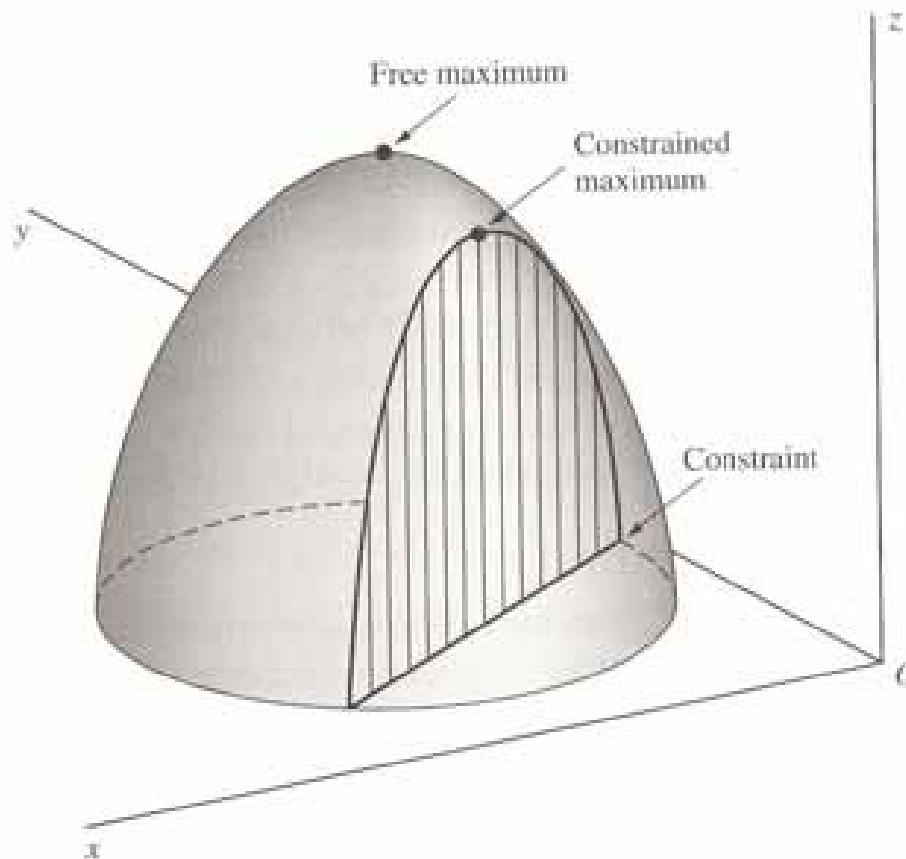
$$\pi = PQ - [C(Q_1) + C(Q_2)]$$

where  $Q = Q_1 + Q_2$ .

- $Q_1$  and  $Q_2$  can be chosen freely in order to maximize profit.  
→ Here,  $Q_1^*$  and  $Q_2^*$  are *free optimal values*.
- Question: If there is a *production constraint*, e.g.  $Q_1 + Q_2 = 900$ , then how would the firm change its behavior?
- →  $Q_1^{**}$  and  $Q_2^{**}$  will now become the *constrained optimal values*.

# Effects of a Constraint

- **Figure:** “Free extremum” vs. “Constrained extremum”



# FINDING THE STATIONARY VALUES

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# Two Choice Variables with One Equality Constraint

➤ Use *elimination* (or *substitution*) method

**Example:**  $\max_{x_1, x_2} U = x_1 x_2 + 2x_1$   
subject to  $4x_1 + 2x_2 = 60$

# Two Choice Variables with One Constraint

➤ Use *Lagrange-multiplier* method

Consider the following maximization problem.

$$\begin{aligned} \max_{x,y} f(x,y) \\ \text{subject to } g(x,y) = c \end{aligned}$$

Define the **Lagrangian function** by

$$L(x, y, \lambda) = f(x, y) + \lambda[c - g(x, y)]$$

F.O.N.C.

$$L_x(x, y) = f_x(x, y) - \lambda g_x(x, y) = 0$$

$$L_y(x, y) = f_y(x, y) - \lambda g_y(x, y) = 0$$

$$L_\lambda(x, y) = c - g(x, y) = 0$$

## Example 1: 2 choice variables & 1 constraint

- Find the optimal values for the following function.

$$U = x_1 x_2 + 2x_1$$

$$\text{subject to } 4x_1 + 2x_2 = 60$$

Step 1- Set up the Lagrangian function:

$$L(x_1, x_2, \lambda) = x_1 x_2 + 2x_1 + \lambda[60 - 4x_1 - 2x_2]$$

Step 2 - Find FONC:

## Example 2: 2 choice variables & 1 constraint

- Find the optimal values for the following function:

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$\text{subject to } g(x_1, x_2) = x_1 + 4x_2 = 2$$

Step 1 - Set up the Lagrangian function:

$$L(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda[2 - x_1 - 4x_2]$$

Step 2 - Find FONC:

# Interpretation of Lagrange Multiplier

- Consider the problem

$$\max_{x,y} (\min) f(x, y) \text{ subject to } g(x, y) = c$$

- Solutions to this problem:  $x^*(c)$ ,  $y^*(c)$ , and  $f^*(c)$ .
  - $f^*(c)$  is called **optimal value** function.
- We can show that  $\frac{df^*(c)}{dc} = \lambda(c)$ .

(See proof next page.)

- The **Lagrange multiplier**  $\lambda$  is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant,  $c$ .

Proof that  $\frac{df^*(c)}{dc} = \lambda(c)$ .

$$\mathcal{L}(c) = f[x(c), y(c)] + \lambda(c)[c - g(x(c), y(c))]$$

$$\frac{d\mathcal{L}(c)}{dc} = \left( f_x \frac{dx}{dc} + f_y \frac{dy}{dc} \right) + \lambda(c) \left[ 1 - g_x \frac{dx}{dc} + g_y \frac{dy}{dc} \right] + [c - g(x, y)] \frac{d\lambda}{dc}$$

$$\frac{d\mathcal{L}(c)}{dc} = (f_x - \lambda g_x) \frac{dx}{dc} + (f_y - \lambda g_y) \frac{dy}{dc} + \lambda(c) + [c - g(x, y)] \frac{d\lambda}{dc}$$

At the optimal point,  $f_x - \lambda^* g_x = f_y - \lambda^* g_y = c - g(x^*, y^*) = 0$ .

Hence,  $\frac{d\mathcal{L}^*(c)}{dc} = \lambda^*(c)$ .

Moreover, since at the optimal point  $c - g(x^*, y^*) = 0$ ,

$$\frac{d\mathcal{L}^*(c)}{dc} = \frac{df[x^*(c), y^*(c)]}{dc}$$

Thus,  $\frac{d\mathcal{L}^*(c)}{dc} = \lambda^*(c) = \frac{df[x^*(c), y^*(c)]}{dc}$ .

# Interpretation of Lagrange Multiplier

- In economic applications,  $c$  denotes the available stock of some resources,  $f(x, y)$  denotes utility or profit.
- $\lambda(c)dc$  is the change in utility or profit that can be obtained from  $dc$  units more of the resource.
- Thus,  $\lambda$  is called a “shadow price” of the resource.

- Examples:

- For cost minimization problem,  $\lambda = \frac{dC}{dQ_0} = MC$
- For utility maximization problem,  $\lambda = \frac{dU}{dY_0} = \text{MU of income}$
- For profit maximization problem,  $\lambda = \frac{d\pi}{dQ_0}$
- For output maximization problem,  $\lambda = \frac{dQ}{dC_0}$

# SECOND-ORDER CONDITIONS FOR CONSTRAINED OPTIMIZATION

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# Second-Order Conditions for Constrained Optimization

- Constraint function:  $g(x, y) = c$

$$dg = g_x dx + g_y dy = 0 \rightarrow dy = -\frac{g_x}{g_y} dx$$

Also,  $d^2g = g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2 + g_y d^2y = 0$

$$\rightarrow d^2y = -\frac{g_{xx}}{g_y} dx^2 - 2\frac{g_{xy}}{g_y} dxdy - \frac{g_{yy}}{g_y} dy^2 \quad (1)$$

- Objective function:  $z = f(x, y)$

$$dz = f_x dx + f_y dy \text{ where } dy = -\frac{g_x}{g_y} dx$$

$$\rightarrow d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 + f_y d^2y$$

$$\rightarrow d^2z = \left(f_{xx} - \frac{g_{xx}}{g_y}\right) dx^2 + 2\left(f_{xy} - 2\frac{g_{xy}}{g_y}\right) dxdy + \left(f_{yy} - \frac{g_{yy}}{g_y}\right) dy^2 \quad (2)$$

# Second-Order Total Differential

- Recall the FONC for  $L(x, y, \lambda) = f(x, y) + \lambda[c - g(x, y)]$

$$\mathcal{L}_x(x, y) = f_x(x, y) - \lambda g_x = 0$$

$$\mathcal{L}_y(x, y) = f_y(x, y) - \lambda g_y = 0$$

- By partially differentiating the FONC, the second derivatives are:

$$\mathcal{L}_{xx} = f_{xx} - \lambda g_{xx} = f_{xx} - \frac{f_y}{g_y} g_{xx}$$

$$\mathcal{L}_{yy} = f_{yy} - \lambda g_{yy} = f_{yy} - \frac{f_y}{g_y} g_{yy}$$

$$\mathcal{L}_{xy} = f_{xy} - \lambda g_{xy} = f_{xy} - \frac{f_y}{g_y} g_{xy}$$

From (2),  $d^2z = \mathcal{L}_{xx}dx^2 + 2\mathcal{L}_{xy}dxdy + \mathcal{L}_{yy}dy^2$

$$\rightarrow d^2z = [dx \quad dy] \begin{bmatrix} \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

# Second-Order Conditions

- The **second-order sufficient conditions** are:
  - For **maximum** of  $z$ :  **$d^2z$  is negative definite**, subject to  **$dg = 0$**
  - For **minimum** of  $z$ :  **$d^2z$  is positive definite**, subject to  **$dg = 0$**
 where  **$dg = g_x dx + g_y dy = 0$** .
- Define the determinant of the **Bordered Hessian matrix** as:

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix}$$

where  $g_x = \partial g / \partial x$ ;  $g_y = \partial g / \partial y$ ;

and  $L_{xx}$ ,  $L_{xy}$ ,  $L_{yx}$ ,  $L_{yy}$  are the second-order derivatives of  $L(x, y, \lambda)$ .

# Second-Order Conditions

- The **sign definiteness of  $d^2z$**  can be determined from the following criterion:\*
- **$d^2z$  is negative definite** subject to  $g_x dx + g_y dy = 0$  iff  $|\bar{H}| > 0$ .
- **$d^2z$  is positive definite** subject to  $g_x dx + g_y dy = 0$  iff  $|\bar{H}| < 0$ .

## • Summary:

For a stationary value of  $L(x, y) = f(x, y) + \lambda[c - g(x, y)]$

The **second-order sufficient conditions** are:

For *maximum* of  $z$ , the ***bordered Hessian is positive*** ( $|\bar{H}| > 0$ );

For *minimum* of  $z$ , the ***bordered Hessian is negative*** ( $|\bar{H}| < 0$ ).

\*See additional notes on the determinantal tests for the sign definiteness of  $d^2z$ .

# Example

- Determine whether the extremum of the objective function

$$U = x_1x_2 + 2x_1 \quad \text{subject to} \quad 4x_1 + 2x_2 = 60$$

gives a minimum or a maximum.

# Example

- Determine whether the extremum of the objective function

$$f(x_1, x_2) = x_1^2 + x_2^2 \quad \text{subject to} \quad g(x_1, x_2) = x_1 + 4x_2 = 2$$

gives a minimum or a maximum.

# ECONOMIC APPLICATIONS OF CONSTRAINED OPTIMIZATION

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# Utility Maximization Problem

- Objective function:  $\max_{Q_1, Q_2} U = U(Q_1, Q_2)$
- Subject to:  $Y_0 = p_1 Q_1 + p_2 Q_2$

➤ Lagrangian function:

$$L(Q_1, Q_2, \lambda) = U(Q_1, Q_2) + \lambda[Y_0 - p_1 Q_1 - p_2 Q_2]$$

➤ F.O.N.C.

$$L_1 = U_1(Q_1, Q_2) - \lambda p_1 = 0$$

$$L_2 = U_2(Q_1, Q_2) - \lambda p_2 = 0$$

$$L_\lambda = Y_0 - p_1 Q_1 - p_2 Q_2 = 0$$

➔  $(Q_1^*, Q_2^*)$  is such that  $\frac{U_1(Q_1, Q_2)}{U_2(Q_1, Q_2)} = \frac{p_2}{p_1}$  and  $Y_0 = p_1 Q_1 + p_2 Q_2$ .

# Utility Maximization Problem

- Second-order sufficient condition

$$|\overline{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & U_{11} & U_{12} \\ p_2 & U_{21} & U_{22} \end{vmatrix} = -p_1^2 U_{22} + 2p_1 p_2 U_{12} - p_2^2 U_{11} > 0$$

# Utility Maximization Problem

- **Example:** Suppose that  $U = 50A + 200B - 0.5A^2 - 2.5B^2$ .  
Find A and B that maximize the utility given that  $P_A = 10$ ,  $P_B = 5$ ,  
and  $Y = 490$ .

# Expenditure Minimization

- Objective function:  $\underset{Q_1, Q_2}{\text{Min}} E = p_1 Q_1 + p_2 Q_2$
- Subject to:  $\bar{U} = U(Q_1, Q_2)$

➤ Lagrangian function:

$$L(Q_1, Q_2, \lambda) = p_1 Q_1 + p_2 Q_2 + \lambda[\bar{U} - U(Q_1, Q_2)]$$

➤ F.O.N.C.

$$L_1 = p_1 - \lambda U_1(Q_1, Q_2) = 0$$

$$L_2 = p_2 - \lambda U_2(Q_1, Q_2) = 0$$

$$L_\lambda = \bar{U} - U(Q_1, Q_2) = 0$$

$(Q_1^*, Q_2^*)$  where

$$\frac{U_1(Q_1, Q_2)}{U_2(Q_1, Q_2)} = \frac{p_1}{p_2} \quad \text{and} \quad \bar{U} = U(Q_1, Q_2)$$

➤ S.O.S.C.  $|\bar{H}| < 0$

# Expenditure Minimization

- **Example:** Suppose that  $U = 50A + 200B - 0.5A^2 - 2.5B^2$ , and the consumer would like to have a utility fixed at  $U_0$ . Find A and B that minimize the expenditure given that  $P_A = 10$  and  $P_B = 5$ .

# Profit Maximization Problem

- Objective function:  $\max_{Q_1, Q_2} \pi = p_1 Q_1 + p_2 Q_2 - c(Q_1, Q_2)$
- Subject to:  $\bar{Q} = Q_1 + Q_2$

➤ Lagrangian function:

$$L(Q_1, Q_2, \lambda) = p_1 Q_1 + p_2 Q_2 - c(Q_1, Q_2) + \lambda[\bar{Q} - Q_1 - Q_2]$$

➤ F.O.N.C.

$$L_1 = p_1 - c_1(Q_1, Q_2) - \lambda = 0$$

$$L_2 = p_2 - c_2(Q_1, Q_2) - \lambda = 0$$

$$L_\lambda = \bar{Q} - Q_1 - Q_2 = 0$$

$(Q_1^*, Q_2^*)$  where

$$p_1 - c_1 = p_2 - c_2 \text{ and } \bar{Q} = Q_1 + Q_2$$

➤ S.O.S.C.  $|\bar{H}| > 0$

# Profit Maximization Problem

- **Example:** Suppose that  $P_1 = 80$ ,  $P_2 = 100$ ,  $TC = 100 + 0.1Q_1^2 + 0.2Q_2^2$   
And  $Q_1 + Q_2 = 325$ . Find  $Q_1$  and  $Q_2$  that maximize the profit.

# Output Maximization Problem

- Objective function:  $\max_{K,L} Q = Q(K, L)$
- Subject to:  $\bar{C} = wL + rK$

➤ Lagrangian function:

$$L(K, L, \lambda) = Q(K, L) + \lambda[\bar{C} - wL - rK]$$

➤ F.O.N.C.

$$L_K = Q_K(K, L) - \lambda r = 0$$

$$L_L = Q_L(K, L) - \lambda w = 0$$

$$L_\lambda = \bar{C} - wL - rK = 0$$

$(K^*, L^*)$  where

$$\frac{Q_K(K, L)}{Q_L(K, L)} = \frac{r}{w} \quad \text{and} \quad \bar{C} = wL + rK$$

➤ S.O.S.C.  $|\bar{H}| > 0$

# Output Maximization Problem

- **Example:** Suppose  $Q = KL$ ,  $w = 6$ ,  $r = 10$ ,  $C_0 = 60$ . Find  $K$  and  $L$  that maximizes output.

# Cost Minimization

- Objective function:  $\underset{K,L}{\text{Min}} C = wL + rK$
- Subject to:  $\bar{Q} = f(K,L)$
- Lagrangian function:

$$L(K,L,\lambda) = wL + rK + \lambda[\bar{Q} - Q(K,L)]$$

- F.O.N.C.

$$L_K = r - \lambda f_K(K,L) = 0$$

$$L_L = w - \lambda f_L(K,L) = 0$$

$$L_\lambda = \bar{Q} - f(K,L) = 0$$

$(K^*, L^*)$  where

$$\frac{f_K(K,L)}{f_L(K,L)} = \frac{r}{w} \quad \text{and} \quad \bar{Q} = f(K,L)$$

- S.O.S.C.  $|\bar{H}| < 0$

# Cost Minimization

- Graph: Expansion path
- ➔ Expansion path describes the least cost combination of  $K^*$  and  $L^*$  required to produce varying levels of quantities.

# Cost Minimization

- **Example:** Suppose  $Q = KL$ . A firm will produce 15 units of the good. Find  $K$  and  $L$  that minimize total cost given that  $w = 6$ ,  $r = 10$ .

# EXTENSION: N-CHOICE VARIABLE AND MULTI-CONSTRAINT CASES

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# Optimization with Equality Constraints: *n*-Variable and One Equality Constraint

- The objective function

$$z = f(x_1, x_2, \dots, x_n)$$

Subject to the constraint  $g(x_1, x_2, \dots, x_n) = c$

## The Lagrangian function:

$$L(x_1, \dots, x_n, \lambda) = f(x_1, \dots, x_n) + \lambda[c - g(x_1, \dots, x_n)]$$

## FONC:

$$L_1 = f_1(x_1, \dots, x_n) - \lambda g_1(x_1, \dots, x_n) = 0$$

⋮

$$L_n = f_n(x_1, \dots, x_n) - \lambda g_n(x_1, \dots, x_n) = 0$$

$$L_\lambda(x_1, \dots, x_n) = c - g(x_1, \dots, x_n) = 0$$

# Second-Order Conditions: $n$ -Variable and One Equality Constraint

- The objective function  $z = f(x_1, x_2, \dots, x_n)$   
Subject to  $g(x_1, x_2, \dots, x_n) = c$

➤ **Bordered Hessian:**

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}$$

- **Bordered leading principal minors** can be defined as:

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}; |\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}; \dots; |\bar{H}_n| = |\bar{H}|$$

(Note: The subscript refers to the size of the bordered matrix.)

# Second-Order Conditions: *n*-Variable and One Equality Constraint

- The conditions for positive and negative definiteness of  $d^2z$  are:

$$d^2z \text{ is } \begin{cases} \text{positive\_definite} \\ \text{negative\_definite} \end{cases} \text{ subject to } dg = 0 \text{ iff } \begin{cases} |\bar{H}_2|, |\bar{H}_3|, \dots, |\bar{H}_n| < 0 \\ |\bar{H}_2| > 0; |\bar{H}_3| < 0; |\bar{H}_4| > 0; \text{etc.} \end{cases}$$

Condition	Maximum	Minimum
First-order necessary condition	$L_\lambda = L_1 = L_2 = \dots = L_n = 0$	$L_\lambda = L_1 = L_2 = \dots = L_n = 0$
Second-order necessary condition*	$ \bar{H}_2  > 0;  \bar{H}_3  < 0;$ $ \bar{H}_4  > 0; \dots; (-1)^n  \bar{H}_n  > 0$	$ \bar{H}_2 ,  \bar{H}_3 , \dots,  \bar{H}_n  < 0$

# Cost Minimization

- Objective function:  $\underset{K,L,T}{\text{Min}} C = wL + iK + rT$
- Subject to:  $Q_0 = f(K, L, T)$
- Lagrangian function:

# Optimization with Equality Constraints: $n$ -Variable and Two Equality Constraints

- The objective function

$$z = f(x_1, x_2, \dots, x_n)$$

Subject to the constraints  $g(x_1, x_2, \dots, x_n) = c$

$$h(x_1, x_2, \dots, x_n) = d$$

## ➤ The Lagrangian function:

$$L = f(x_1, x_2, \dots, x_n) + \lambda_1 [c - g(x_1, x_2, \dots, x_n)] + \lambda_2 [d - h(x_1, x_2, \dots, x_n)]$$

## ➤ FONC:

$$L_i = f_i(x_1, \dots, x_n) - \lambda_1 g_i(x_1, \dots, x_n) - \lambda_2 h_i(x_1, \dots, x_n) = 0 \quad \text{for } i = 1, \dots, n$$

$$L_{\lambda_1}(x_1, \dots, x_n) = c - g(x_1, \dots, x_n) = 0$$

$$L_{\lambda_2}(x_1, \dots, x_n) = d - h(x_1, \dots, x_n) = 0$$

# Optimization with Equality Constraints: $n$ -Variable and $k$ Equality Constraints

- The objective function

$$z = f(x_1, x_2, \dots, x_n)$$

Subject to the constraints

$$g_1(x_1, x_2, \dots, x_n) = c_1$$

$$g_2(x_1, x_2, \dots, x_n) = c_2$$

.....

$$g_k(x_1, x_2, \dots, x_n) = c_k$$

## ➤ The Lagrangian function:

$$L = f(x_1, x_2, \dots, x_n) + \lambda_1 [c_1 - g^1(x_1, x_2, \dots, x_n)] + \lambda_2 [c_2 - g^2(x_1, x_2, \dots, x_n)]$$

$$+ \dots + \lambda_k [c_k - g^k(x_1, x_2, \dots, x_n)]$$

# Optimization with Equality Constraints: $n$ -Variable and $k$ Equality Constraints

➤ **FONC:**

$$\frac{\partial L}{\partial x_1} = f_1(x_1, \dots, x_n) - \lambda_1 g_1^1(x_1, \dots, x_n) - \dots - \lambda_k g_1^k(x_1, \dots, x_n) = 0$$

$$\frac{\partial L}{\partial x_2} = f_2(x_1, \dots, x_n) - \lambda_1 g_2^1(x_1, \dots, x_n) - \dots - \lambda_k g_2^k(x_1, \dots, x_n) = 0$$

.....

$$\frac{\partial L}{\partial x_n} = f_n(x_1, \dots, x_n) - \lambda_1 g_n^1(x_1, \dots, x_n) - \dots - \lambda_k g_n^k(x_1, \dots, x_n) = 0$$

$$\frac{\partial L}{\partial \lambda_1} = c_1 - g^1(x_1, \dots, x_n) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = c_2 - g^2(x_1, \dots, x_n) = 0$$

.....

$$\frac{\partial L}{\partial \lambda_k} = c_k - g^k(x_1, \dots, x_n) = 0$$

# Second-Order Conditions: $n$ -Variable and $k$ Equality Constraints

The Lagrangian condition is:

$$L = f(x_1, x_2, \dots, x_n) + \sum_{j=1}^k \lambda_j [c_j - g^j(x_1, x_2, \dots, x_n)]$$

➤ **Bordered Hessian:**

$$|\overline{H}| \equiv \begin{array}{c|cccc|cccc} 0 & 0 & \cdots & 0 & g_1^1 & g_2^1 & \cdots & g_n^1 \\ 0 & 0 & \cdots & 0 & g_1^2 & g_2^2 & \cdots & g_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g_1^k & g_2^k & \cdots & g_n^k \\ \hline g_1^1 & g_1^2 & \cdots & g_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2^1 & g_2^2 & \cdots & g_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n^1 & g_n^2 & \cdots & g_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{array}$$

# Second-Order Conditions: $n$ -Variable and $k$ Equality Constraints

- The second-order sufficient conditions are stated in terms of the signs of the following  $(n-k)$  bordered leading principal minors:

$$|\overline{H}_{k+1}|, |\overline{H}_{k+2}|, \dots, |\overline{H}_n| \quad \text{where } |\overline{H}_n| = |\overline{H}|$$

- SOSC:

- For a *maximum of  $z$* , the bordered leading principal minors **alternate in sign**, and the sign of  $|\overline{H}_{k+1}| = (-1)^{k+1}$ .
- For a *minimum of  $z$* , all the bordered leading principal minors **take the same sign**, namely that of  $(-1)^k$ .

# Example

- The objective function

$$z = f(x_1, x_2, x_3)$$

$$\text{Subject to } g(x_1, x_2, x_3) = c$$

$$h(x_1, x_2, x_3) = d$$

- The Lagrangian function:

$$L = f(x_1, x_2, x_3) + \lambda_1 [c - g(x_1, x_2, x_3)] + \lambda_2 [d - h(x_1, x_2, x_3)]$$

- FONC:  $L_1 = L_2 = L_3 = L_{\lambda_1} = L_{\lambda_2} = 0$

- SOSC: 1) Max:  $|H_3| < 0$

$$2) \text{ Min: } |H_3| > 0$$