



EE 320 Introductory Mathematical Economics

Semester 1/2019

Homework 3 - Suggested Answers

Question 1

Calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ (if possible) of the following functions.

a. $f(x, y) = \frac{5xy^2}{x^2+y^2}$

b. $f(x, y) = \ln(x^2y + xy^2) - x^2 - y^2$

c. $f(x, y, z) = xz^2 \ln(y) - \frac{y}{z^2+x-y}$

d. $f(x, y, z) = e^{x+\ln(z)} - \ln(x^2)y^2z^3$

a. $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(5y^2) - (5xy^2)(2x)}{(x^2+y^2)^2}$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(10xy) - (5xy^2)(2y)}{(x^2+y^2)^2}$$

b. $\frac{\partial f}{\partial x} = \frac{2xy+y^2}{x^2y+xy^2} - 2x$

$$\frac{\partial f}{\partial y} = \frac{x^2+2xy}{x^2y+xy^2} - 2y$$

c. $\frac{\partial f}{\partial x} = z^2 \ln(y) + [y/(z^2+x-y)]^2$

$$\frac{\partial f}{\partial y} = xz^2/y - [(z^2+x)/(z^2+x-y)]^2$$



$$\partial f / \partial z = 2xz \ln(y) + [2yz / (z^2 + x - y)^2]$$

d. $\partial f / \partial x = ze^x - (2/x)y^2z^3$

$$\partial f / \partial y = -2y \ln(x^2)z^3$$

$$\partial f / \partial z = e^x - 3z^2 \ln(x^2)y^2$$

Question 2

The optimal profit function of a firm can be given by,

$$\pi^*(p, w_1, w_2, A) = A * p^{\frac{1}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)}}$$

where $0 < \beta < 1$ and $\gamma < 1$, A is the level of technology, P is price of output, w_1 is the factor price of capital, and w_2 is the factor price of labor.

Consider the following problem

- a. Use the partial derivative to conclude about the relationship between price and the level of profit.

$$\partial \pi^* / \partial p = \frac{1}{1-\beta} A * p^{\frac{\beta}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)}} > 0$$

- b. How does the technical progress affect the level of profit? Show your result by using the partial derivative.

$$\partial \pi^* / \partial A = p^{\frac{1}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)}} > 0$$

- c. How does the level of factor price of inputs affect the level of profit? Show your result for both types of input, using the partial derivative. Then, explain the intuition of your result in economics.

$$\partial \pi^* / \partial w_1 = \left[\frac{\beta}{(\beta-1)} \right] A p^{\frac{1}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)}-1} * w_1^{\gamma-1} < 0$$



$$\partial \pi^* / \partial w_2 = \left[\frac{\beta}{\beta - 1} \right] A p^{\frac{1}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)} - 1} * w_2^{\gamma-1} < 0$$

This is because $0 < \beta < 1 \Rightarrow \beta - 1 < 0 \Rightarrow \left[\frac{\beta}{\beta-1} \right] < 0$

- d. Show that the profit function is convex in factor price of inputs. That is, the second-order partial derivatives of profit with respect to factor price of both capital and labor are greater than zero.

$$\begin{aligned} \partial^2 \pi^* / \partial w_1^2 &= \left[\frac{\beta}{\beta - 1} \right] (\gamma - 1) A p^{\frac{1}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)} - 1} * w_1^{\gamma-2} \\ &+ \left[\frac{\beta}{\beta - 1} \right] \left[\frac{\beta}{\gamma(\beta - 1)} - 1 \right] A p^{\frac{1}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)} - 2} * \gamma w_1^{2(\gamma-1)} > 0 \end{aligned}$$

$$\begin{aligned} \partial^2 \pi^* / \partial w_2^2 &= \left[\frac{\beta}{\beta - 1} \right] (\gamma - 1) A p^{\frac{1}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)} - 1} * w_2^{\gamma-2} \\ &+ \left[\frac{\beta}{\beta - 1} \right] \left[\frac{\beta}{\gamma(\beta - 1)} - 1 \right] A p^{\frac{1}{1-\beta}} (w_1^\gamma + w_2^\gamma)^{\frac{\beta}{\gamma(\beta-1)} - 2} * \gamma w_2^{2(\gamma-1)} > 0 \end{aligned}$$

Note that $\left[\frac{\beta}{\beta-1} \right] < 0$, $(\gamma - 1) < 0$ and $\left[\frac{\beta}{\gamma(\beta-1)} - 1 \right] < 0$.

Question 3

Consider a simple macroeconomic model given below,

$$\begin{aligned} Y &= C + G \\ C &= C_0 + c_1(Y - T) \\ T &= T_0 + t_1 Y \\ G &= G_0 \end{aligned}$$

where Y is national income, C is consumption, T is the amount of tax collected, G is the level of government expenditure.

Answer the following questions:



- a. State all the endogenous variables. What are the parameters and exogenous variables in the model?

Endogenous: C, Y, T

Exo and Parameters: C_0, c_1, T_0, t_1, G_0

- b. Derive the equilibrium solution of all the endogenous variables?

$$Y^* = \frac{C_0 - c_1 T_0 + G_0}{1 - (1 - t_1)c_1}$$

$$T^* = T_0 + t_1 \left[\frac{C_0 - c_1 T_0 + G_0}{1 - (1 - t_1)c_1} \right]$$

$$C^* = C_0 + c_1 \left[(1 - t_1) \left(\frac{C_0 - c_1 T_0 + G_0}{1 - (1 - t_1)c_1} \right) - T_0 \right]$$

- c. Use the partial derivative to show the effect change in c_1 and G_0 on the equilibrium of endogenous variables?

$$\begin{aligned} \frac{\partial y^*}{\partial c_1} &= \frac{[1 - (1 - t_1)c_1](-T_0) + (C_0 - c_1 T_0 + G_0)(1 - t_1)}{(1 - (1 - t_1)c_1)^2} \\ &= \frac{(C_0 + G_0)(1 - t_1) - T_0}{(1 - (1 - t_1)c_1)^2} > 0 \end{aligned}$$

$$\frac{\partial y^*}{\partial G_0} = \frac{1}{1 - (1 - t_1)c_1} > 0$$

$$\frac{\partial T^*}{\partial c_1} = t_1 \left[\frac{(C_0 + G_0)(1 - t_1) - T_0}{(1 - c_1(1 - t_1))^2} \right]$$

$$\frac{\partial T^*}{\partial G_0} = \frac{t_1}{1 - (1 - t_1)c_1} > 0$$

$$\frac{\partial C^*}{\partial c_1} = (1 - t_1) \left[\frac{C_0 - c_1 T_0 + G_0}{1 - (1 - t_1)c_1} + c_1 \frac{(C_0 + G_0)(1 - t_1) - T_0}{(1 - (1 - t_1)c_1)^2} \right] - T_0 > 0$$



$$\frac{\partial C^*}{\partial G_0} = \frac{c_1(1-t_1)}{1-(1-t_1)c_1} > 0$$

- d. How does the marginal propensity to tax (t_1) affect the equilibrium level of income (Y^*) and consumption (C^*)?

$$\partial Y^*/\partial t_1 = -\frac{(c_0 - c_1 T_0 + G_0)c_1}{(1 - (1 - t_1)c_1)^2} < 0$$

$$\partial C^*/\partial t_1 = -c_1 \left[\frac{C_0 - c_1 T_0 + G_0}{1 - (1 - t_1)c_1} + (1 - t_1) \frac{c_1(C_0 - c_1 T_0 + G_0)}{(1 - (1 - t_1)c_1)^2} \right] < 0$$

Question 4

Write the Hessian Matrix for each of the following functions:

a. $U(x, y) = 7x^2 + 8xy + 3y^2$

$$\text{Ans. } H = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 8 & 6 \end{bmatrix}$$

b. $z(x, y) = 5(13x - 5y)^2$

$$\text{Ans. } H = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix} = \begin{bmatrix} 1690 & -650 \\ -650 & 250 \end{bmatrix}$$

c. $f(x, y) = 4x^3 - 11xy - 7y^5$

$$\text{Ans. } H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 24x & -11 \\ -11 & -140y^3 \end{bmatrix}$$

d. $Q(K, L) = (2K + 1)(3L^2 + 2)$

$$\text{Ans. } H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LK} & Q_{LL} \end{bmatrix} = \begin{bmatrix} 0 & 12L \\ 12L & 6(2K + 1) \end{bmatrix}$$

Question 5



The demand for a product depends on the price p_1 of the product and on the price p_2 charged by a competing producer, and it is given by:

$$D(p_1, p_2) = 36 - \frac{8p_1}{\sqrt{p_2}}$$

- a. Find $\frac{\partial D}{\partial p_1}$ and $\frac{\partial D}{\partial p_2}$, and comment on the signs of the partial derivatives.

Ans. $\frac{\partial D}{\partial p_1} = -8(p_2)^{-1/2} < 0$, showing that the demand decreases as its own price increases.

$\frac{\partial D}{\partial p_2} = -(8) \left(-\frac{1}{2}\right) p_1 (p_2)^{-3/2} = 4p_1 (p_2)^{-3/2} > 0$, showing that the demand increases as the price of a competing product increases.

- b. Calculate the own-price and cross-price elasticities of demand when $p_1 = 3$ and $p_2 = 4$.

Ans. When $p_1 = 3$ and $p_2 = 4$, $D = 36 - (8)(3)/2 = 24$

$$\varepsilon_{D,p_1} = \frac{\partial D/D}{\partial p_1/p_1} = \frac{\partial D}{\partial p_1} \cdot \frac{p_1}{D} = -\frac{8}{2} \cdot \frac{3}{24} = -0.5$$

$$\varepsilon_{D,p_2} = \frac{\partial D}{\partial p_2} \cdot \frac{p_2}{D} = \frac{12}{8} \cdot \frac{4}{24} = 0.25$$

Question 6

Suppose the production function Q depends on the number of workers L according to the formula:

$$Q = Lg\left(\frac{\ln(L)}{L}\right)$$

where $g(\cdot)$ is a differentiable function. Find expressions for $\frac{dQ}{dL}$ and $\frac{d^2Q}{dL^2}$.

Ans. $\frac{dQ}{dL} = g'(u) \left[\frac{1}{L} - \frac{\ln(L)}{L} \right] + g(u)$ where $u = \frac{\ln(L)}{L}$

$$\frac{d^2Q}{dL^2} = g'(u) \left[-\frac{1}{L^2} - \left(\frac{1}{L} - \frac{\ln(L)}{L} \right) \right] + \left[\frac{1}{L} - \frac{\ln(L)}{L} \right] g''(u) \frac{du}{dL} + g'(u) \frac{du}{dL} = -\frac{g'(u)}{L^2} + \left[\frac{1}{L} - \frac{\ln(L)}{L} \right]^2 g''(u)$$



Question 6: Given the production function $Q = f(K, L) = A[K^n + L^n]$ where A is the level of technology, K is capital and L is labor. Suppose that $n > 0$. Consider the following problem.

- a) To ensure that the above production function exhibits a *decreasing return to scale technology*, what additional restrictions do one need to place on n?
- b) Under the assumption used in (a), show that the production function satisfies the law of diminishing returns.
- c) Calculate the marginal rate of technical substitution (MRTS) of labor (L) for capital (K).
- d) Show that MRTS is a decreasing function in L. That is, as labor increases, the value of MRTS decreases.
- e) Under the condition(s) assumed in (a), does the production function have the *global concave* property? (Optional)

Suppose that $K(t) = \frac{1}{2}t^2 + 2t + 3$ and $L(t) = e^t + 3$, where $t \geq 0$ is the number of periods from now. Consider the following problem

- f) Show that Q is increasing over time.
- g) Compute $\frac{dQ}{dt}$ when $t = 0$, i.e. growth of output in the initial period.

- a) To ensure a decreasing return to scale, n must be less than one.
- b) This requires that MPK decreases in K, and MPL decreases in L.

$$\frac{\partial MPK}{\partial K} = An(n-1)k^{n-2} \rightarrow n < 1 \text{ is needed to ensure the decreasing.}$$

$$\frac{\partial MPL}{\partial L} = An(n-1)l^{n-2} \rightarrow n < 1 \text{ is needed to ensure the decreasing.}$$

$$c) MRTS_{L \text{ for } K} = |dK/dL| = MPL/MPK = (L/K)^{n-1} = (K/L)^{1-n}$$

$$d) \quad \frac{dMRTS_{L \text{ for } K}}{dL} = \{L^{(1-n)}(1-n)K^{(-n)}dK/dL - (1-n)L^{-n}K^{1-n}\} / L^{2(1-n)}$$

Note that $(1-n) > 0$ (from the condition in b) but $dK/dL < 0$.

As a result, $dMRTS_{L \text{ for } K}/dL < 0$.



$$e) H = \begin{bmatrix} An(n-1)K^{n-2} & 0 \\ 0 & An(n-1)L^{n-2} \end{bmatrix}$$

Note that $|H1| = An(n-1)k^{n-2} < 0$ for all positive value of K.

Also, we observe that $|H2| = A^2n^2(n-1)^2(kL)^{n-2} > 0$ for any strict positive combination of K and L. This implies that H is negative definite for all strict positive combination of K and L, and hence suggesting that $f(K,L)$ is globally concave.

$$f) \frac{dQ}{dt} = MPK * \left(\frac{dK}{dt}\right) + MPL * \left(\frac{dL}{dt}\right)$$

Note that $\left(\frac{dK}{dt}\right) = t + 2 > 0$ for all $t \geq 0$

$$\left(\frac{dL}{dt}\right) = e^t > 0 \text{ for all } t \geq 0$$

Since both MPK and MPL are greater than zero, this implies that $\frac{dQ}{dt} > 0$.

That is, output is a strictly increasing function over time.

$$h) \text{ For } t = 0, \text{ we note that } K(0) = 3 \text{ and } L(0) = 4 \rightarrow Q(t=0) = A[3^n + 4^n]$$

$$\frac{dQ}{dt} = MPK(t=0) * (2) + MPL(t=0) * (1)$$

(Use the above equation that we derived $\frac{dK}{dt}$ and $\frac{dL}{dt}$, and substitute $t=0$ into.)

$$MPK(t=0) = AnK(t=0)^{n-1} = An(3^{n-1})$$

$$MPL(t=0) = AnL(t=0)^{n-1} = An(4^{n-1})$$

$$\left(\frac{dQ}{dt}\right)/Q = \{An(3^{n-1})*2 + An(4^{n-1})*1\} / A[3^n + 4^n] \#$$