# Matrices,
# Algebra of Matrices
# &
# Elementary Operations

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## What is a Matrix ? (Not a movie trilogy staring Keanu Reaves)

A matrix is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called the entries (or elements) of the matrix.

For example,

$$\mathbf{A} = [a_{ij}] = \begin{array}{cl} \begin{array}{cccc} \text{Column1} & \text{Column2} & & \text{Column n} \end{array} & \\ \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & . & \cdots & a_{mn} \end{array}\right] & \begin{array}{l} \text{Row1} \\ \text{Row 2} \\ \\ \text{Row m} \end{array} \end{array}$$

**A** is a matrix of dimension (size) *m* x *n* (*m* rows and *n* columns)

$a_{ij}$ is an entry or an element of the matrix

$a_{ii}$ is a main diagonal entry

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$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}$ is a 2 x 3 matrix (Rectangular Matrix)

$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}$ is a 2 x 2 matrix (Square Matrix)

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3 x 3 matrix (Square Matrix)

$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is a diagonal matrix

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$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is an identity matrix or a unit matrix

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ is an upper triagular matrix

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 5 \end{bmatrix}$ is a lower triagular matrix

$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ is a zero matrix

$[a_1 \quad a_2 \quad a_3]$. is a 1 x 3 matrix (we usually call it a row vector)

$\begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$ is a 2 x 1 matrix (we usually call it a column vector)

## Vector

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector. We shall denote vectors by ***lowercase*** boldface letters **a, b,** $\cdots$ or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots, \quad a_n]. \qquad \text{For instance,} \qquad \mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

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## Basic operations of matrices

**Equality of Matrices**

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal,** written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different.** Thus, matrices of different sizes are always different.

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \qquad \text{if and only if} \qquad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{matrix}$$

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**Addition of Matrices**

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of **A** and **B.** Matrices of different sizes cannot be added.

$$\mathbf{A} + \mathbf{B} = \left[ a_{jk} + b_{jk} \right]_{m \times n}$$

For example;

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{bmatrix}$$

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**Scalar Multiplication (Multiplication by a Number)**

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** $c$ (number $c$) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of **A** by $c$.

$$c\mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \quad \text{then} \quad -\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}, \quad \frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} =$$

$$\mathbf{A} + \mathbf{C} = \qquad 2\mathbf{B} =$$

$$\mathbf{A} - 2\mathbf{B}$$

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Properties of Matrix Addition

Cummulative law (a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

Associative law of addition (b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}.$

Properties of Scalar Multiplication

Distributive laws

(a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A} = \mathbf{A}.$

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## Matrix multiplication

Three different ways with the same answer:

Method 1: Each entry of $AB$ is the product of a row and a column.

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \\ \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} \\ & (AB)_{ij} & \\ \\ \end{bmatrix}$$

$(AB)$ij=row i of $A$ times column j of $B$

This single entry is the inner product of the two vectors.

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Example

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$$

**EXAMPLE** $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute $AB$, if it is defined.

*Solution:* Since $A$ is $2 \times 3$ and $B$ is $3 \times 2$, then $AB$ is defined and $AB$ is ______×______.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

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**EXAMPLE:** If $A$ is $4 \times 3$ and $B$ is $3 \times 2$, then what are the sizes of $AB$ and $BA$?

*Solution:*

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} \phantom{*} & \phantom{*} \\ \phantom{*} & \phantom{*} \\ \phantom{*} & \phantom{*} \\ \phantom{*} & \phantom{*} \end{bmatrix}$$

$BA$ would be $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$ Which is ____________

$$\boxed{\text{If } A \text{ is } m \times n \text{ and } B \text{ is } n \times p\text{, then } AB \text{ is } m \times p.}$$

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**Method 2**: Each column of $AB$ is the product of a matrix and a column

Suppose $A$ is $m \times n$ and $B$ is $n \times p$ where $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

$$A_{m\times n}B_{n\times p} = \begin{bmatrix} A_{m\times n}\begin{bmatrix} b_{11} \\ b_{21} \\ . \\ . \\ b_{n1} \end{bmatrix} & A_{m\times n}\begin{bmatrix} b_{12} \\ b_{21} \\ . \\ . \\ b_{n2} \end{bmatrix} & . \; . & A_{m\times n}\begin{bmatrix} b_{1p} \\ b_{2p} \\ . \\ . \\ b_{np} \end{bmatrix} \end{bmatrix}$$

Column j of $AB$ = $A$ times column j of $B$

The number of columns in $A$ has to equal the number of rows in $B$.

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Example $\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$

Example

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} =$$

**EXAMPLE:** Compute $AB$ where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

*Solution:*

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of $A$ and $A\mathbf{b}_2$ is a linear combination of the columns of $A$.

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Method 3: Each row of $AB$ is the product of a row and a matrix

row i of $AB$ = row i of $A$ times $B$

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} =$$

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## Properties of Matrix Multiplication

**Cautions !**

**AB** is <u>not always</u> equal to **BA**

Try

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

If **AB** = **AC** , **B** is not necessary equal to **C**

eg.

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \text{and } C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

$B \neq C$ but $AB = AC$

If **AB** = **0** , **A** or **B** is not necessary equal to **0**

eg.

$$A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \qquad AB = 0$$

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## Properties of matrix multiplication

Let $A$ be $m \times n$ and let $B$ and $C$ have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B+C) = AB + AC$ (left - distributive law)

c. $(B+C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$
for any scalar $r$

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

**Transposition of Matrices and Vectors**

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^{\mathsf{T}}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on.

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & . & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{A}^T = \left[a_{ji}\right] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

**Transposition of Matrices and Vectors**

If $\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}$, then $\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$.

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**EXAMPLE:** Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute $AB$, $(AB)^T$, $A^TB^T$ and $B^TA^T$.

*Solution:*

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \qquad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \qquad \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \qquad \end{bmatrix}$$

$$\boxed{(AB)^T = \qquad}$$

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## Properties of Matrix Transposition

Let $A$ and $B$ denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of $A^T$ is $A$)

b. $(A+B)^T = A^T + B^T$

c. For any scalar $r$, $(rA)^T = rA^T$

d. $(AB)^T = B^TA^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order. )

**EXAMPLE:** Prove that $(ABC)^T =$ _________.

*Solution:* By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T(\qquad)^T$$

$$= C^T(\qquad) = \_\_\_\_\_\_\_\_\_.$$

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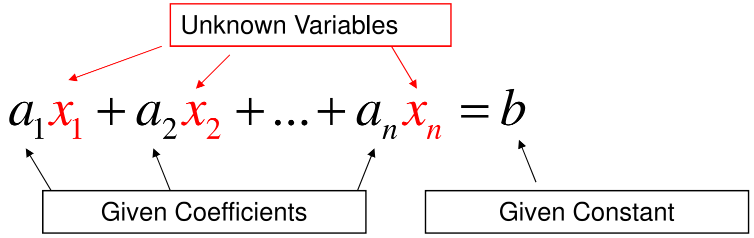
# Solution of System of Linear Equations

# System of Linear Equations

A linear equation



Unknown Variables

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Given Coefficients

Given Constant

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A linear equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \qquad (*)$$

eg.

$$2x_1 - x_2 + 5x_3 = 2\sqrt{5}$$

Non-linear equation

Anything that is not in the form of a linear equation (*)

eg.

$$2x_1 - x^2{}_2 + 5x_3\sin(x_1) = 25$$

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# System of Linear Equations

A collection of one or more linear equations involving the same set of variables.

A system of linear equations with 2 variables:

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned} \qquad \text{E.g.} \quad \begin{aligned} 2x + y &= 8 \\ x + 3y &= 9 \end{aligned}$$

A system of linear equations with 3 variables:

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + 2x_3 &= 4 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$

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The whole idea of linear algebra is to solve

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

A system of linear equations can be written in matrix form

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + 2x_3 &= 4 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$$\begin{bmatrix} 6 & 1 & 1 \\ 5 & 1 & 2 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix} \qquad \text{or} \qquad \left[\begin{array}{ccc|c} 6 & 1 & 1 & 6 \\ 5 & 1 & 2 & 4 \\ 4 & 1 & -1 & -2 \end{array}\right]$$

An augmented matrix form

$$[\mathbf{A}|\mathbf{b}]$$

A matrix equation: $\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$

## A system of linear equations with n variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots a_{3n}x_n &= b_3\\
. \qquad\qquad\qquad\qquad &\quad .\\
. \qquad\qquad\qquad\qquad &\quad . \qquad\qquad (1)\\
. \qquad\qquad\qquad\qquad &\quad .\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots a_{nn}x_n &= b_n
\end{aligned}$$

$x_1, x_2, x_3, \ldots\ldots, x_n$ Is a set of unknown variables

If <u>all $b_i$ are zero</u> then the system is called <u>"Homogeneous system"</u>

If $b_i$ <u>are not all zero</u> then the system is called <u>"Non homogeneous system"</u>

**If the system (1) is homogeneous, it has at least the trivial solution $x_1=0,\ldots,x_n=0$**

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## Solutions of System of Linear Equations

**A linear system of $m$ equations in $n$ unknowns $x_1, \cdots, x_n$ is a set of equations of the form**

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2\\
\cdots\cdots\cdots&\cdots\cdots\cdots\\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Can be written in a form of matrix equation as

$$\mathbf{Ax} = \mathbf{b} \longrightarrow [\mathbf{A}|\mathbf{b}]$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}}_{m \times n}\underbrace{\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}}_{m \times 1}$$

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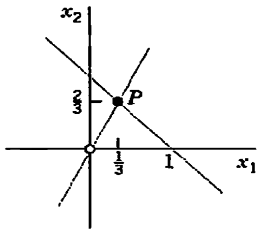
## Geometric Interpretation. Existence and Uniqueness of Solutions

If $m = n = 2$, we have two equations in two unknowns $x_1, x_2$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1\\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

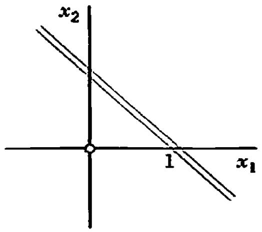
There are 3 possible cases

$x_1 + x_2 = 1$
$2x_1 - x_2 = 0$
Case (*a*)



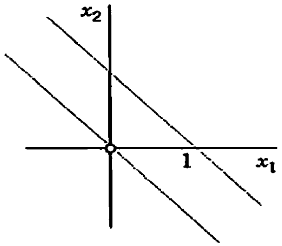
(*a*) Precisely one solution if the lines intersect.

$x_1 + x_2 = 1$
$2x_1 + 2x_2 = 2$
Case (*b*)



(*b*) Infinitely many solutions if the lines coincide.

$x_1 + x_2 = 1$
$x_1 + x_2 = 0$
Case (*c*)



(*c*) No solution if the lines are parallel

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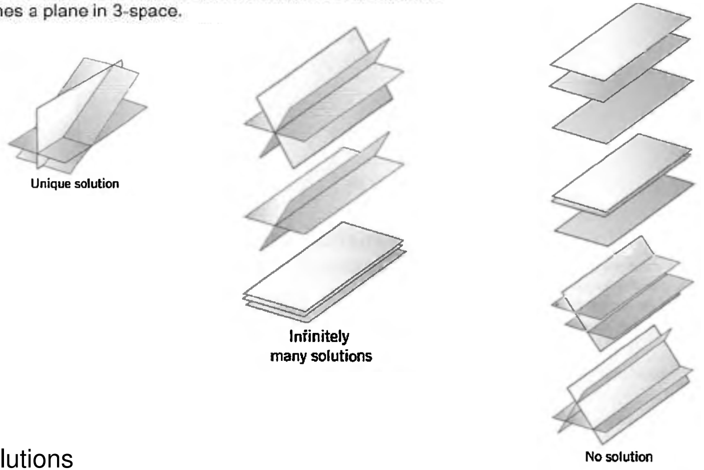
**EXAMPLE:** Three equations in three variables. Each equation determines a plane in 3-space.



<u>Solutions</u>

Unique or Infinitely many solutions ⟶ Consistent system

No solution ⟶ Inconsistent system

**How can we know ?**

## Strategy for solving a linear system

Replace one system with **an equivalent system** (one with the same solution set) that is easier to solve.

example

a)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

b)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$

c)
$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

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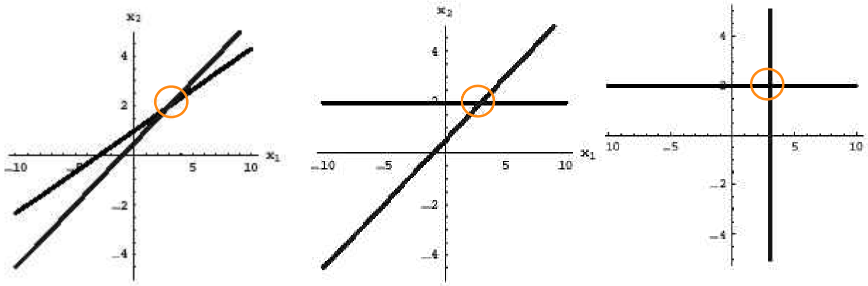
## Equivalent systems



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

a)

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$

b)

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

c)

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## Solving a linear system

**Elementary Row Operations:**

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Note: **Row equivalent matrices:** Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

**Fact about Row Equivalence:** If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

**Replacement**: *k* x Row *i* adds to Row *j* and **replace Row *j* (the one that is not multiplied.)**

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**Example** Solving a system of linear equations using augmented matrix methods.

$$3x_1 + 4x_2 = 1$$

$$x_1 - 2x_2 = 7$$

1. Augmented matrix corresponding to the system of linear equations.

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

2. $R_1 \leftrightarrow R_2$ (To get a 1 in the upper left corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

3. $(-3)R_1 + R_2 \to R_2$ (To get a 0 in the lower left corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

4. $\left(\frac{1}{10}\right)R_2 \to R_2$ (To get a 1 in the lower right corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

5. $(2)R_2 + R_1 \to R_1$ (To get a 0 in the upper right corner.)

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

Hence, $x_1 = 3$ and $x_2 = -2$.

---

We can stop the row operation process at step 4 and perform back substitution to obtain the solution set. This method is known as "**Gauss Elimination**" method.

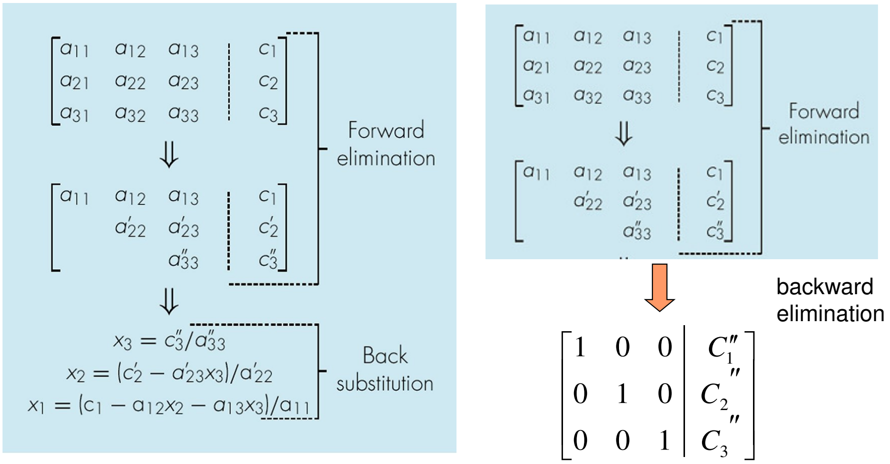
4. $\left(\frac{1}{10}\right)R_2 \to R_2$ (To get a 1 in the lower right corner.)

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

$$x - 2y = 7 \qquad (1)$$
$$y = -2 \qquad (2)$$

Solve for y first in eq. (2)
and then substitute y into eq.(1)
to solve for x

---



$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ & a'_{22} & a'_{23} & c'_2 \\ & & a''_{33} & c''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$
$$x_3 = c''_3/a''_{33}$$
$$x_2 = (c'_2 - a'_{23}x_3)/a'_{22}$$
$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

Back substitution

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$
$$\Downarrow$$
$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ & a'_{22} & a'_{23} & c'_2 \\ & & a''_{33} & c''_3 \end{array}\right]$$

Forward elimination

backward elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & C''_1 \\ 0 & 1 & 0 & C''_2 \\ 0 & 0 & 1 & C''_3 \end{array}\right]$$

---

Example of system of linear equations with 3 variables

An augmented matrix

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$R_2+4R_1 \to R_2$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$R_2/2$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$R_3+3R_2 \to R_3$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

R₂+4R₃ ➔| R₂

R₁+2R₂ ➔ R₁

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

---

$$\begin{array}{rcl} x_1 & = & 29 \\ x_2 & = & 16 \\ x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

**Check:** Is $(29, 16, 3)$ a solution of the *original* system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcll} (29)-2(16)+3 & = & 29-32+3 & = 0 \\ 2(16)-8(3) & = & 32-24 & = 8 \\ -4(29)+5(16)+9(3) & = & -116+80+27 & = -9 \end{array}$$

---

## Row operations are reversible.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution sets.

Example 1

$$6x_1 + x_2 + x_3 = 6$$
$$5x_1 + x_2 + 2x_3 = 4$$
$$4x_1 + x_2 - x_3 = -2$$

$$x_1 = 3, x_2 = -13, x_3 = 1$$

---

Example 2

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$x_1 = -1 + \frac{4}{3}x_3$$

$$x_2 = 2$$

## Example 3

$$3x_2 - 6x_3 = 8$$
$$x_1 - 2x_2 + 3x_3 = -1$$
$$5x_1 - 7x_2 + 9x_3 = 0$$

Inconsistent

## Example 4

$$2x_1 - 4x_2 + x_3 = -4$$
$$4x_1 - 8x_2 + 7x_3 = 2$$
$$-2x_1 - 4x_2 - 3x_3 = 5$$

Inconsistent

## Example 5

Example 5 : For what values of $h$ will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

**Solution:** Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{array}\right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

## Example 7

Example 7 Give an example of **b** that will make the linear system consistent.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 &= b_2 \\ 3x_1 + 6x_2 + 8x_3 &= b_3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 2 & 4 & 6 & b_2 \\ 3 & 6 & 8 & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & 2 & b_3 - 3b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & -b_2 + 2b_1 + b_3 - 3b_1 \end{array}\right]$$

For consistent system 0 = b3-b2-b1

Eg. $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

**Echelon form** (or **row echelon form**):

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

**EXAMPLE:** Echelon forms

(a) $$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

The whole purpose of doing the elimination is to get an upper triangular matrix (U) (echelon form)

---

**Reduced echelon form:** Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

**EXAMPLE** (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

---

### Important terms

**Pivot position**: a position of a leading entry in an echelon form of the matrix.

**Pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's

**Pivot column**: a column that contains a pivot position.

**Pivot row**: a row that contains a pivot position.

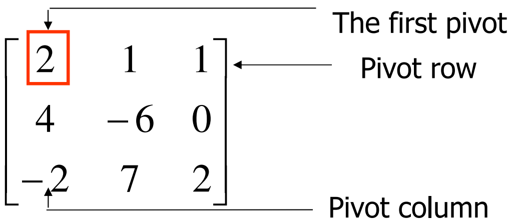
Number of pivots = **Rank of a matrix**

Example : 3 equations 3 unknowns

(coefficient matrix)

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

---



$$\begin{bmatrix} \boxed{2} & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Replacement $$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 8 & 3 \end{bmatrix} \xrightarrow{\text{Replacement}} \underset{U}{\begin{bmatrix} \boxed{2} & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 0 & \boxed{1} \end{bmatrix}}$$

By definition, pivots cannot be zero. ( need to divide them!)

Example Row reduce to echelon form and locate the pivot columns.

pivot

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \underset{R_1 \leftrightarrow R_4}{\sim} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \underset{\substack{R_2+R_1 \rightarrow R_2 \\ R_3+2R_1 \rightarrow R_3}}{\sim} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

pivot column

Possible Pivots:

$$\underset{\substack{R_3-(5/2)R_2 \rightarrow R_3 \\ R_4+(3/2)R_2 \rightarrow R_4}}{} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Original Matrix:

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$\begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \\ 1 & 2 & & 4 & \end{array}$$

pivot columns: 1 2 4

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

---

**EXAMPLE:** Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \underset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$R_2-R_1 \rightarrow R_2 \quad \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

---

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \overset{R_2/2}{\sim} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\underset{R_3-3R_2 \rightarrow R_3}{\sim} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \underset{R_2-R_3 \rightarrow R_2}{\overset{R_1-6R_3 \rightarrow R_1}{}} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Echelon form

$$\underset{R_1+9R_2 \rightarrow R_1}{\sim} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \overset{R_1/3}{\sim} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Reduced echelon form

---

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

How many pivot variables?

How could this 3x3 case fail to give answer?

$$\begin{bmatrix} \bigcirc\!\!\!\!* & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

-Fail to come up with 3 pivots.

Example

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & \bigcirc\!\!\!\!0 & -2 \\ 0 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

If this entry is changed to 0
→ matrix would not have been invertible

---

# The breakdown of elimination

Under what circumstances could the elimination break down?

If the algorithm produces *n* pivots (a full set of pivots),
then there is only one solution to the equations (nonsingular case)

If a zero appears in a pivot position, elimination has to stop!
Stop temporarily → there is possibility to exchange with a lower row for a proper pivot.
Stop permanently → there is no exchange of row that can avoid zero.

However, we do not know whether a zero will appear until we try.

---

## Two Fundamental Questions (Existence and Uniqueness)

1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

If a solution exists either a unique solution or infinitely many solutions , the system is said to be consistent. Otherwise the system is inconsistent.

How an echelon form (or rref) of a matrix (obtained by performing Row operations on a matrix ) tell us about the nature of the solution (e.g. Unique solution, infinitely many solutions or no solution)?

---

**EXAMPLE:** Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

In the last example, this system was reduced to the triangular form:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is sufficient to see that the system is consistent and unique. Why?

**EXAMPLE:** Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

**Solution:** Row operations produce:

$$\left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \quad \leftarrow \textit{Never true} \end{aligned}$$

The original system is inconsistent!

---

**EXAMPLE:** For what values of $h$ will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

**Solution:** Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{array}\right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

---

example

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 8 \\ 4 & 14 & 16 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

m-n=2
These 2 rows are linear combination of other rows

(a) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 8 \\ 3 & 12 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Unique solutions

(b) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 8 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 5/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5/3 \\ 0 & 0 & 0 \end{bmatrix}$ No solutions

(c) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 6 \\ 5 & 10 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Many solutions

---

- Solution to Ux=0 is the same as solution to Ax=0
- Number of pivot variables (basic variables ) = r
- Number of free variables = n-r

$$\boxed{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \enclose{circle}{x_2} \\ x_3 \\ \enclose{circle}{x_4} \end{bmatrix} = 0}$$

- •*Assign any number to X2 and X4*
- •*Solve for X1 and X3*

$$\Rightarrow \boxed{\begin{aligned} x_1 &= -2x_2 - 2x_3 - 2x_4 \\ x_3 &= -2x_4 \end{aligned}}$$

Parametric description of general solutions

## Example

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 2 & 4 & 6 & 8 & 5 \\ 3 & 6 & 8 & 10 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Ans $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

---

## Example

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

General solutions

$$x_1 = -1 + \frac{4}{3} x_3$$

$$x_2 = 2$$

---

## Example

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \quad \& \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix}$$

RREF

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 6 & 1 & 7 \\ 5 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -5 & -5 \\ 0 & -17 & -17 \\ 0 & -14 & -14 \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

r=n <m
(full column rank)

---

## Solutions of linear systems

Pivot variable (Basic variable) :
➔any variables that corresponds to a pivot column

Free variable: all nonbasic variables

**EXAMPLE:**

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \qquad \begin{array}{rl} x_1 + 6x_2 \quad + 3x_4 & = 0 \\ x_3 - 8x_4 & = 5 \\ x_5 & = 7 \end{array}$$

Pivot columns:

Pivot variables:

Free variables

Final step in solving a consistent linear system: After the augmented matrix is in reduced echelon form and the system is written down as a set of equations

Solve each equation for the pivot variables in terms of the free variables (if any) in the equations

**EXAMPLE:**

$$\begin{array}{ccccccc} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array} \qquad \begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

(general solution)

The system is consistent and has infinite many solutions

---

**EXAMPLE:**

$$\left[\begin{array}{cccccc} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{array}\right]$$

In an earlier example, we obtained the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is consistent.

**Free variables:** $x_3$ and $x_4$

$\boxed{\textbf{Consistent system with free variables}}$ ⇒ **infinitely many solutions.**

---

**EXAMPLE:**

$$\begin{array}{rrcr} 3x_1 & +4x_2 & = & -3 \\ 2x_1 & +5x_2 & = & 5 \\ -2x_1 & -3x_2 & = & 1 \end{array} \rightarrow \left[\begin{array}{rr|r} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rr|r} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

$\boxed{\textbf{Consistent system, no free variables}}$ Full column rank ⇒ **unique solution.** $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$

---

# Inverse of Matrices

# Matrix Inverses

The inverse of a real number $a$ is denoted by $a^{-1}$. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix $A$ is said to be **invertible** if there is an $n \times n$ matrix $C$ satisfying

$$CA = AC = I_n$$

where $I_n$ is the $n \times n$ identity matrix. We call $C$ the **inverse** of $A$.

example $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ ; $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ $AC=$ $CA=$

The inverse of $A$ is usually denoted by $A^{-1}$.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

***Not all $n \times n$ matrices are invertible.*** A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

---

Fact 1 If $A$ is invertible, then the inverse is unique.

*Proof:* Assume $B$ and $C$ are both inverses of $A$. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_\_)\_\_\_\_\_\_ = I\_\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.■

Fact 2 The inverse of $A^{-1}$ is A itself.

Fact 3 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then $A$ is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then $A$ is not invertible.

---

Assume $A$ is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$\_\_\_\_\_A\mathbf{x} = \_\_\_\_\_\mathbf{b}$ and so

$I\mathbf{x} = \_\_\_\_\_\_\_\_$ or $\mathbf{x} = \_\_\_\_\_\_\_\_$.

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$\_\_\_\_\_A\mathbf{w} = \_\_\_\_\_\mathbf{b}$ which means $\mathbf{w} = A^{-1}\mathbf{b}$.

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Fact 4 If $A$ is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

---

**EXAMPLE:** Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{array}{rcrcl} -7x_1 & + & 3x_2 & = & 2 \\ 5x_1 & - & 2x_2 & = & 1 \end{array}.$$

*Solution:* Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

## Properties of Inverses

Suppose A and B are invertible. Then the following results hold:

a. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$ (i.e. $A$ is the inverse of $A^{-1}$).

b. $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$

*Partial proof of part b:*

$$(AB)(B^{-1}A^{-1}) = A(\_\_\_\_\_\_\_\_\_)A^{-1}$$

$$= A(\_\_\_\_\_\_\_\_\_\_)A^{-1} = \_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

*Proof part c*

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

---

## Matrix inversion algorithm

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

➔Take a column at a time, that equation determines the columns of $A^{-1}$

$A$ times column j of $A^{-1}$ = column j of $I$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

Carry out elimination on
*all systems simultaneously.*

---

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$[A \mid I]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$[I \mid A^{-1}]$$

The Gauss-Jordan method

---

## Matrix inversion algorithm

Place $A$ and $I$ side-by-side to form an augmented matrix $[A \mid I]$. Then perform row operations on this matrix (which will produce identical operations on $A$ and $I$).

$[A \mid I]$ will row reduce to $[I \mid A^{-1}]$

or $A$ is not invertible.

**EXAMPLE:** Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

*Solution:*

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists

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# Determinant of Matrices

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**Determinants**

Determinant of a square matrix **A** is denoted by $|\mathbf{A}|$ or det **A** is a uniquely defined SCALAR associated with that matrix. Determinants are defined only for square matrices

*A second-order determinant*

For a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

is a scalar

Example : Given

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 5 & 9 \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = (6)(9) - (-3)(5) = 69$$

---

*A Third-order determinant*

For a 3x3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}) - (a_{13}a_{31}a_{22} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

or

$$\det \mathbf{A} = |\mathbf{A}| = \left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right.$$

OR

Subdeterminant or Minor

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \qquad [= \text{a scalar}]$$

---

## *A Third-order determinant*

Example :

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (2)(5)(9) + (1)(6)(7) + (3)(8)(4) - (2)(8)(6) - (1)(4)(9) - (3)(5)(7) = -9$$

$$\begin{vmatrix} -7 & 0 & 3 \\ 9 & 1 & 4 \\ 0 & 6 & 5 \end{vmatrix} = (-7)(1)(5) + (0)(4)(0) + (3)(6)(9) - (-7)(6)(4) - (0)(9)(5) - (3)(1)(0) = 295$$

---

## *An n^th^ -order determinant*

For an *n x n* matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \left[a_{ij}\right]_{n\times n}$$

By expanding

Any row $i$ $\quad \det \mathbf{A}$ or $|\mathbf{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$

Any column $j$ $\quad \det \mathbf{A}$ or $|\mathbf{A}| = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$

$C_{ij}$ is the “cofactor” of the element

$M_{ij}$ is the “minor” of the element $a_{ij}$

Obtained by deleting the *i* th row and *j* th column of a given determinant

---

The **(i, j)-cofactor** of $A$ is the number $C_{ij}$ where

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$\det A = a_{11}.c_{11} + a_{12}c_{12} + \ldots\ldots a_{1n}c_{1n}$$

A cofactor expansion across the first row of $A$

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example

Evaluate determinant of **A**, given

$$\mathbf{A}=\begin{bmatrix}0&1&2\\1&3&4\\5&6&7\end{bmatrix}$$

*Choose row 1 for expansion, since there is 0 in row 1*

$$\det\mathbf{A}=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$$

$$C_{12}=(-1)^{1+2}(M_{12})\rightarrow M_{12}=\begin{vmatrix}1&4\\5&7\end{vmatrix}=(1)(7)-(4)(5)=-13$$

$$C_{13}=(-1)^{1+3}(M_{13})\rightarrow M_{13}=\begin{vmatrix}1&3\\5&6\end{vmatrix}=(1)(6)-(3)(5)=-9$$

$$\det\mathbf{A}=(1)(3)+(2)(-9)=-5$$

**EXAMPLE:** Compute the determinant of $A=\begin{bmatrix}1&2&0\\3&-1&2\\2&0&1\end{bmatrix}$ using cofactor expansion down column 3.

*Solution*

$$\begin{vmatrix}1&2&0\\3&-1&2\\2&0&1\end{vmatrix}=0\begin{vmatrix}3&-1\\2&0\end{vmatrix}-2\begin{vmatrix}1&2\\2&0\end{vmatrix}+1\begin{vmatrix}1&2\\3&-1\end{vmatrix}=1.$$

Example

$$A=\begin{bmatrix}1&5&0\\2&4&-1\\0&-2&0\end{bmatrix}\qquad \det A=?$$

(-2)

**EXAMPLE:** Compute the determinant of $A=\begin{bmatrix}1&2&3&4\\0&2&1&5\\0&0&2&1\\0&0&3&5\end{bmatrix}$

(14)

Basic properties of determinants

(1) $\det\mathbf{A}^T=\det\mathbf{A}$

***Example 1*** $\begin{vmatrix}4&3\\5&6\end{vmatrix}=\begin{vmatrix}4&5\\3&6\end{vmatrix}=9$

***Example 2*** $\begin{vmatrix}a&b\\c&d\end{vmatrix}=\begin{vmatrix}a&c\\b&d\end{vmatrix}=ad-bc$

**Hence, column operations = row operations in determinant. (2)-(4)**

(2) The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value of the determinant.

*Example* $\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc.\qquad \begin{vmatrix}c&d\\a&b\end{vmatrix}=cb-ad=-(ad-bc)$

$$\begin{vmatrix}0&1&3\\2&5&7\\3&0&1\end{vmatrix}=-26.\qquad \begin{vmatrix}3&1&0\\7&5&2\\1&0&3\end{vmatrix}=26.$$

(3) The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold.

*Example*

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} 15a & 7b \\ 12c & 2d \end{vmatrix} = 3\begin{vmatrix} 5a & 7b \\ 4c & 2d \end{vmatrix} = 3(2)\begin{vmatrix} 5a & 7b \\ 2c & d \end{vmatrix} = 6(5ad - 14bc)$$

---

(4) The addition (subtraction) of a multiple of any row (or column) to (from) another row will leave the value of the determinant unaltered.

*Example*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = a(d + kb) - b(c + ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

---

**EXAMPLE:** Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

*Solution*

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

---

**EXAMPLE:** Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

(-12)

**EXAMPLE:** Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

(-10)

---

Compute Det(**A**)

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

Det(**A**) = 0 when **A** is not invertible (singular).

(0)

---

(5) If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

*Example*

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0 \qquad \begin{vmatrix} c & c \\ d & d \end{vmatrix} = cd - cd = 0$$

(6) A zero row or column renders the value of a determinant zero.

*Example*

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 = \begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 8 \\ 0 & 3 & 9 \end{vmatrix}$$

---

Further Properties

(7) $$\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}$$

**EXAMPLE:** Compute $\det A^3$ if $\det A = 5$.

*Solution:* $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_.$

(8) If **A** is an $n$ x $n$ upper or lower triangular matrix

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

$$\det \mathbf{A} = a_{11}a_{22}a_{33}\ldots a_{nn}$$

Example

$$[A] = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 10 & 0 \\ 4 & 8 & 9 \end{bmatrix} \rightarrow |A| = (2)(10)(9) \ \mathbf{= 180}$$

---

## Applications of determinants

### Cramer's Rule

Cramer's rule can be used to study how the solution of **Ax=b** affected by the changes in the entries of b.

Let $A$ be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $R^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1,2,3 \ldots n$$

where

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \ \ . \ \ . \ \ \underset{\substack{\uparrow \\ Col\ i}}{\mathbf{b}} \ \ . \ \ . \ \ \mathbf{a}_n]$$

---

Example

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

---

Find the solution of the equation system

$$\begin{aligned} 7x_1 - \ x_2 - \ x_3 &= 0 \\ 10x_1 - 2x_2 + \ x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$

## The computation of $A^{-1}$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \tag{4}$$

The matrix of cofactors on the right side of (4) is called the **adjugate** (or **classical adjoint**) of $A$, denoted by adj $A$. (The term *adjoint* also has another meaning in advanced texts on linear transformations.) The next theorem simply restates (4).

The adjoint of $A_{nxn}$ is defined to be the transpose of the matrix of cofactors:

$$\boxed{adjA = \left[C_{ij}(A)\right]^T}$$

$$adjA ==> n \times n$$

---

Example

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 4 & 7 & 3 \\ -2 & 8 & 5 \end{bmatrix}$$

$$adjA = \begin{bmatrix} 11 & -26 & 46 \\ -8 & 13 & -24 \\ 7 & -13 & 21 \end{bmatrix}^T = \begin{bmatrix} 11 & -8 & 7 \\ -26 & 13 & -13 \\ 46 & -24 & 21 \end{bmatrix}$$

Theorem

**An Inverse Formula**

Let $A$ be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\,\text{adj}\,A$$

---

**EXAMPLE 3** Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

**Solution** The nine cofactors are

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

The adjugate matrix is the *transpose* of the matrix of cofactors. [For instance, $C_{12}$ g in the (2, 1) position.] Thus

---

$$\text{adj}\,A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

We could compute det $A$ directly, but the following computation provides a check the calculations above *and* produces det $A$:

$$(\text{adj}\,A)\cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

Since $(\text{adj}\,A)A = 14I$, Theorem 8 shows that det $A = 14$ and

$$A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$